Solitary wave in a homogeneous Bose-Einstein condensate

A Thesis submitted by

Elaf Salah Mousa

under the supervision of

Dr/Ashraf A. Abul Seoud.

Abstract

Solitary wave solutions of the Gross Pitaevskii equation which describe the motion of a Bose Einstein condensed (BEC) atoms are obtained.

General expressions are derived for the velocity and the spatial profile of the solitary waves.

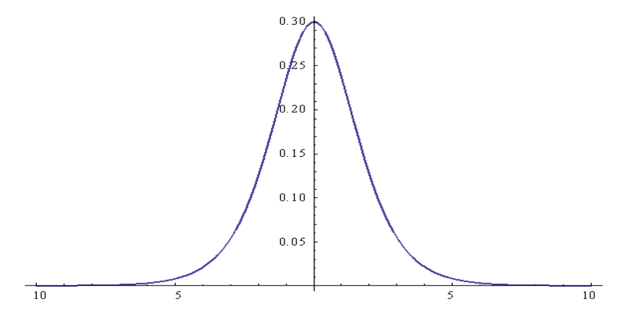
Contentntns

- Solitons and Solitary waves .
- The Ideal Bose-Einstein Condensation .
- Solitary wave in homogeneous Bose gas .

Solitons and Solitary waves

Solitary wave :

Often occur as a single entity



1-It has a permanent form.
2-It is localized in space.
3-it`s energy is finite.

The first solitary wave was observed by Scott Russel .

And has wave profile :

$$Z = \zeta(x,t)$$

Where :

$$\zeta(x,t) = a\,\mathrm{sech}^2[\beta(x - ct)]$$

with

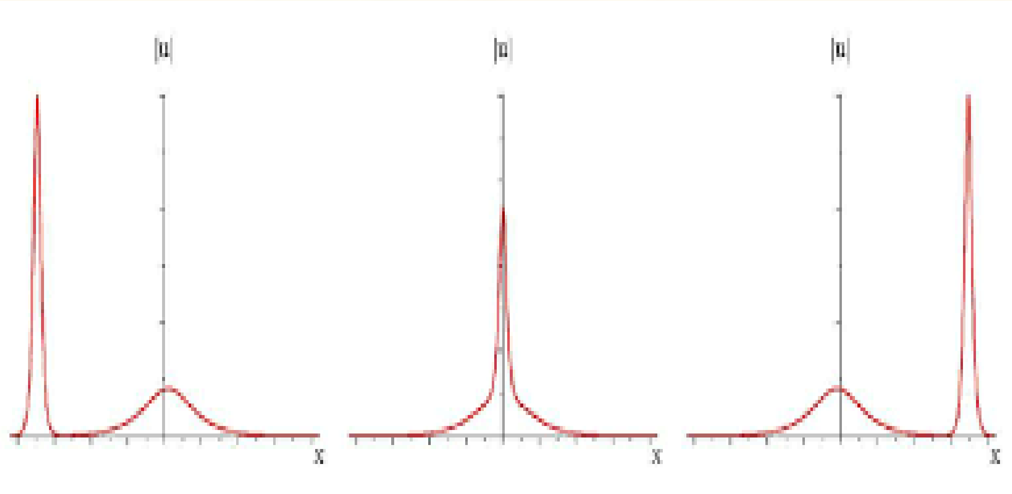
$$\beta^2 = \frac{3a}{4h^2(h+a)}$$

Solitons are waves that have:

- 1- Represent a permanent form.

  2- They are localized in space.
  3- They have finite energy.

  4- They can interact strongly with each other and retain their identity.



The solitary wave solutions:

Are solutions of nonlinear partial differential equations which contains

- Nonlinear terms.
- Dispersion terms.

The effect of these terms cancell each other and therefor we obtain waves of permanent form.

There are several methods to compute exact, analytic expressions for solitary and periodic waves, They include :

- The direct integration method.
- The inverse spectral transform method.
- Lie-Backlund transformation method

In the direct integration method

We seek travelling wave solutions. By working in a travelling frame of reference the PDE is replaced by an ordinary differential equation (ODE) for which one seeks closed-form solutions .

Bose-Einstein Condensation

Bose atoms obey the Bose-Einstein distribution:

$$F(\varepsilon) = \frac{1}{\exp \beta(\varepsilon - \mu) - 1} \qquad \beta = \frac{1}{k_B T}$$

and the total number of particles is :

$$N = \sum_{k=0}^{\infty} \frac{1}{z^{-1} e^{\beta \varepsilon_k} - 1}$$

where z is the (Fugacity) = $e^{\beta\mu}$

The particles of the system either occupy the ground or the excited states.

$$N = N_o + N_T$$

Where

$$N_o = \frac{1}{z^{-1} - 1}$$

and

$$N_T = \sum_{k \neq 0} \frac{1}{z^{-1} e^{\beta \varepsilon_k} - 1}$$

In the Thermodynamic limit, the number of particles out of the condensate is given by :

$$N_T = \frac{V}{\lambda^3} g_{\frac{3}{2}}(z)$$

where $g_n(z)$ is the bosonic function given by :

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx$$

Critical Temperature T_C :

On cooling the gas to lower temperatures the value of Z gradually increases until it eventually equals one. At this point the Chemical Potential μ becomes zero .

The temperature at which this occurs is called the critical temperature.

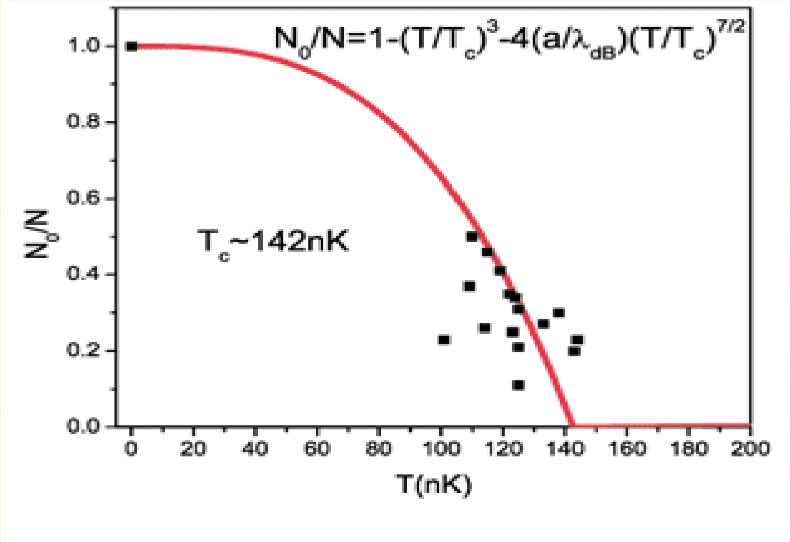
Einstein realized that as soon as this happen the number of particles in the lowest energy quantum state becomes infinite .

So that there is a finite fraction of all of the particles $\frac{N_o}{N}$ are in the one quantum state .

Which is related to the criticle temperature by

$$\frac{N_o}{N} = 1 - \left(\frac{T}{T_C}\right)^{\frac{3}{2}}$$

The behaviour of the condensate fraction as a function of the ratio $\frac{T}{T_C}$ is shown in the figure below.



- **The non-linear schrodinger equation**

The bose atoms interact with each other through a delta function , and they obey the nonlinear Schrodinger equation

$$(-\frac{\hbar^2}{2m}\nabla^2 + V(r)_{eff})\psi_i(r) = \varepsilon_i \psi_i(r)$$

where

$$V_{eff}(\vec{r}) = U_0 \delta(\vec{r} - \vec{r}')$$

Solitary wave in homogeneous Bose gas

The condensate wave function satisfies the nonlinear Schrodinger equation :

$$(-\frac{\hbar^2}{2m}\nabla^2 + U_o\,|\psi|^2)\psi_i(r) = i\hbar\frac{\partial\psi}{\partial t}$$

where $U_0 = \frac{4\pi\hbar^2 a_{sc}}{m}$
is the effective two-body interaction.

m is the atomic mass,

and a is the scattering length for atom-atom collisions.

We write the condensate wave function in the form :

$$\psi = \sqrt{n}e^{i\phi}$$

where n is the particle density, which is spatially uniform in equilibrium and in the absence of an external potential ,and ϕ is the phase of the wave function.

working in terms of a superfluid velocity $\bar{v} = \frac{\hbar}{m}\vec{\nabla}\phi$

We replaced the nonlinear Schrodinger equation by differential equations for "n" and ϕ using :

$$\frac{\partial\psi}{\partial t} = \left(\frac{\partial\sqrt{n}}{\partial t} + i\,\frac{\partial\phi}{\partial t}\sqrt{n}\right)e^{i\phi}$$

And

$$\vec{\nabla}^2\psi = (\{\vec{\nabla}^2\sqrt{n} - \sqrt{n}\,|\vec{\nabla}\phi|^2\} + i\{(\vec{\nabla}^2\phi)\sqrt{n} + 2(\vec{\nabla}\phi)(\vec{\nabla}\sqrt{n})\})e^{i\phi}$$

Substituting in the nonlinear Schrodinger equation and equating the real and imaginary parts of the equation we obtain:

the equation of continuity

$$\frac{\partial n}{\partial t} + \vec{\nabla}.(n\vec{v}) = 0$$

Finally we obtained :

$$m\frac{\partial\vec{v}}{\partial t}=\vec{\nabla}\{\frac{\hbar^2}{2m}\frac{1}{\sqrt{n}}\vec{\nabla}^2\sqrt{n}-\frac{1}{2}mv^2-\mu(n)\}$$

Here we introduced the Chemical Potential for a dilute Bose gas .

$$\mu(n)=nU_o$$

The chemical potential shows how much work we have to do to stick a new particle in the system.

We look for solutions of the equations for which the fluid velocity and particle density propagate at a uniform velocity u, without change of form.

$$U = f(z - ut)$$

$$n = g(z - ut)$$

As $Z \to \infty$ $n = n_0$

To obtain solutions that are localized In space, n must satisfy the condition

$$n_{\min} \leq n \leq n_o$$

The solitary wave solutions correspond to an oscillation of the particle from $n = n_0$ to $n = n_{\min}$ and back again. They are depressions in the density.

And the velocity of the wave is equal to the sound speed at the minimum density

The solution depends on the coherence length corresponding to the background density

$$\xi(n_0) = (8\pi n_0 a_{sc})^{\frac{-1}{2}}$$

also the tanh function was introduced

$$\chi = \tanh^2\left(\frac{z}{\zeta} + c\right)$$

where

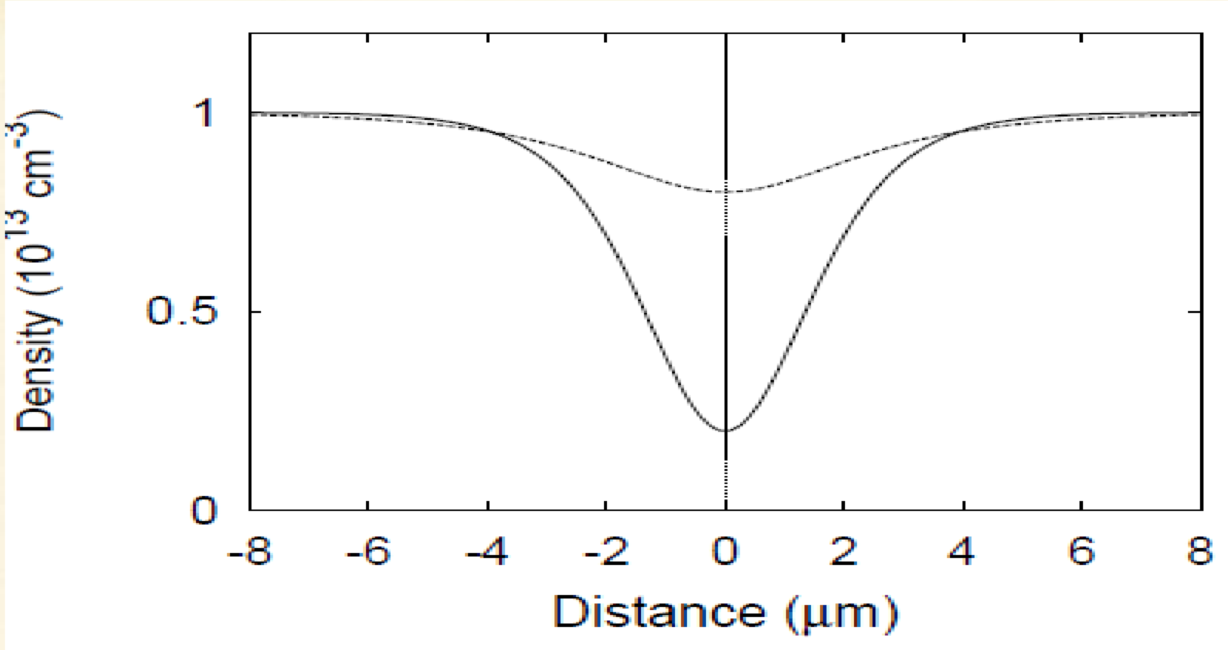
$$\zeta = 2^{\frac{1}{2}} \xi(n_0)\left(1 - \frac{n_{\min}}{n_o}\right)^{\frac{-1}{2}}$$

We choose the coordinate of the system such that : c=0

Thus

$$n(z) = n_{\min} + (n_0 - n_{\min}) \tanh^2(\tfrac{z}{\zeta})$$

ζ which gives the spatial extent of the solitary wave, is on the order of the coherence length that corresponds to the background density.



This figure shows the profile of two solitary waves, for (Na)atoms with a background density of :

$$n_0 = 10^{13}\,cm^{-3}$$

and for two values of : $\frac{n_{\min}}{n_0} \sim 20\%$

(solid curve) and 80%(dashed curve).

Finally, we point out that If U=0

Thus $n_{\min} = 0$ then we get the well known solution :

$$\psi = \sqrt{n_0}\,\tan\left(\frac{z}{\zeta}\right)$$

which is sometimes referred to as a (Dark Soliton) .

Refrences

- Paper by: A. D. Jackson, G. Kavoulakis, and C. J. Pethick . (1998). Phys. Rev. 58,

- James F. Annett Superconductivity, Superfluids, and Condensates Oxford Master Series in Condensed Matter Physics 5 2004

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Questions ?

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Thank You ☺