On The Construction of Three Step Derivative Free Four-Parametric Methods with Accelerated Order of Convergence

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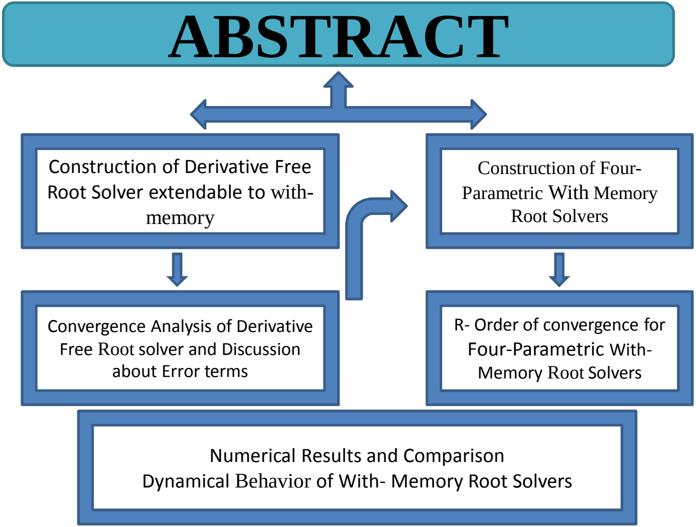
Abstract


ABSTRACT
Construction of Derivative Free Root Solver extendable to with-memory
Construction of Four-Parametric With Memory Root Solvers
Convergence Analysis of Derivative Free Root solver and Discussion about Error terms
R- Order of convergence for Four-Parametric With-Memory Root Solvers
Numerical Results and Comparison
Dynamical Behavior of With- Memory Root Solvers

Iterative Schemes Using Derivatives and Their Drawbacks

In 1960 and following three decades iterative schemes with derivative are more in focus like Newton's method , Jarrat type methods, King's Method and many more. These schemes have following drawbacks:
1. Derivative are not availible frquently, their computational cost is also questionable.
2. Their convergence deponds on the choice of initial guess. The method does not converge if starting point lie in the vicinity of root.
Due to this, algorithms in which no derivative evaluation is needed are more in concentration. This approach was firstly addressed by Steffensen and later used by Alicia Cordero, Petkovic, Neta and many other researchers.

With and Without Memory Root Solvers

Multi-step root solvers that use only information from the recent iteration are called without memory root solvers and the root finding methods that use information from the recent and previous iteration are known as with-memory iterative root solvers. Traub defined multipoint iteration function with memory as:
" Let z_j represents $p+1$ quantities $x_j, w_1(x_j), ..., w_p(x_j), (j \geq 1)$. If x_{k+1} is calculated iteratively by

$$x_{k+1} = \mathsf{F}\left(z_k, z_{k-1}, ..., z_{k-p}\right)$$

then F is called multipoint iteration function with memory.

Importance of Multistep With-Memory Root Solver

Among all discussed methods, multi-step with-memory root solvers are of more significance because they significantly improve the convergence speed and computational efficiency of the without memory root solvers without using any additional functional evaluations. Generally with-memory root solvers are constructed by using one or more free parameters or self-accelerators in any optimal derivative free without memory root solver.

Techniques Used For Conversion to With-Memory Root Solvers

The construction of with-memory optimal root solvers are based on two techniques;

- By the use of inverse interpolation

Motivation and Background

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- Convergence of root solver depends on the choice of intial guess.
- By the use of self accelerating parameters
- Freedom of the choice of initial guess regarding the convergence of root solver.
- These techniques had been applied by Petković and Džunić [12].

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2. What are the preferable functions for such type of acceleration?
3. If Newtonian polynomials are used what is the maximum no. of points to pass?

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- The degree of Newtonian polynomials are dependent on the number of availible nodes from the current and previous iterate.

Efficiency Index Of Some Recently developed With-Memory Methods

Cordero et al.[2] presented a two-parametric with-memory family of two-steps methods based on a without memory fourth order method of Zheng having R-order of convergence at least 7 and its index of efficiency is 1.913.

$$
\begin{aligned}
w_m &= x_m + \mathrm{b}_m f(x_m), \mathrm{b}_m = -\frac{1}{N_3'(x_m)}, \mathrm{g}_m = -\frac{N_4''(w_m)}{2N_4'(w_m)}, m \geq 1, \\
y_m &= x_m - \frac{f(x_m)}{f[x_m, w_m] + \mathrm{g}_m f(w_m)}, \\
x_{m+1} &= y_m - \frac{f(y_m)}{f[x_m, y_m] + (y_m - x_m) f[x_m, w_m, y_m]},
\end{aligned} \tag{1}
$$

Efficiency Index Of Some Recently developed With-Memory Methods

Lotfi et al. in [10] presented a new tri-parametric with-memory method based on without memory two-step variant of Steffensen's method. It is demonstrated that the R-order of convergence of (1) is at least 7.77200 and the efficiency index is $7.77200 \approx 1.98082$.

Optimal Derivative Free Method v/s With-Memory Root Solver

Cordero and Torregrosa in [3] developed a three-step two-parametric derivative free method by using the approximation $f'(x_m) \approx f[x_m, z_m]$, where $z_m = x_m + \mathrm{g} f(x_m)^4$ in the three-step iterative method of Sharma et al. [14], which is given by:

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f[x_m, z_m]}, \quad m \geq 0 \\
u_{m+1} &= y_m - \frac{f(x_m) + \mathrm{b} f(y_m)}{f(x_m) + (\mathrm{b}-2) f(y_m)} \frac{f(y_m)}{f[x_m, z_m]}, \\
x_{m+1} &= x_m - \frac{(P+Q+R) f(x_m)}{Pf[u_m, x_m] + Qf[z_m, x_m] + Rf[y_m, x_m]},
\end{aligned} \tag{2}
$$

where $P = (x_m - y_m)f(x_m)f(y_m)$, $Q = (y_m - u_m)f(y_m)f(u_m)$ and $R = (u_m - x_m)f(u_m)f(x_m)$ with error equation

$$e_{m+1} = c_2^2((1 + 2\mathrm{b})c_2^2 - c_3)(c_2^3 - 2c_3c_2 + c_4)e_m^8 + O(e_m^9). \tag{3}$$

Obviously the error equation (3) cannot allows to improve the convergence order of (2) by varying the free parameters g and b. Thus (2) cannot be extended to with-memory root solver.

Construction of Derivative Free Root Solver Extendable to With-Memory

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3. We use weight function approach at the second step of any optimal two-step with-derivative method followed by Newton's method in the third step.
4. The first derivative arising at each step is calculated using suitable approximations.
5. In this way the convergence order of the proposed methods is preserved and can be increased by varying the involved free parameters.

Construction of Derivative Free Root Solver extendable to with-memory

$$
\begin{aligned}
y_m &= f_1(x_m), m \geq 0, \\
z_m &= f_2(x_m, y_m),
\end{aligned} \tag{4}
$$

where f_1 and f_2 are real functions such that f_1 is the well known Newtons's scheme which involves the values $f(x_m)$ and $f'(x_m)$ providing the quadratic convergence of the sequence x_m and f_2 requires the previously computed values $f(x_m)$, $f'(x_m)$ and the new value $f(y_m)$ to give the fourth order convergence.

Construction of Derivative Free Root Solver extendable to with-memory

Based on any two-step with-derivative optimal method like (4), we design a general three-step without-derivative method extendable to with-memory.

- We use a weight function $S(u_m)$ in the second step of (4) and add another real function f_3, (Newton's method [11]) in the third step.
- The values of first derivative involved in f_1, f_2 and f_3 are approximated by

$$\begin{aligned}
\mathrm{f}_1 &= f[x_m, w_m] + qf(w_m) \\
\mathrm{f}_2 &= f[x_m, w_m] + qf(w_m), f[y_m, w_m] + qf(w_m) + s(y_m - w_m)(y_m - \\
\mathrm{f}_3 &= f[y_m, z_m] + f[z_m, y_m, x_m](z_m - y_m) + f[z_m, y_m, x_m, w_m](z_m - y \\
&\quad (z_m - x_m) + t(z_m - w_m)(z_m - y_m)(z_m - x_m)
\end{aligned}$$

where $u_m = \frac{f(y_m)}{f(x_m)}$ and the scalars p, q, s and t are freely chosen parameters. Hence, we obtain the following general three-step method .

Construction of Derivative Free Root Solver extendable to with-memory

$$
\begin{aligned}
y_m &= \mathrm{f}_1(x_m, w_m), w_m = x_m + pf(x_m), m \geq 0, \\
z_m &= \mathrm{f}_2(x_m, w_m, y_m), \\
x_{m+1} &= \mathrm{f}_3(x_m, w_m, y_m, z_m),
\end{aligned}
\tag{5}
$$

were f_1 is the famous Steffensen's method [17] employing the values $f(x_m)$ and $f(w_m)$, f_2 is selected such that it requires the previously calculated values $f(x_m)$, $f(w_m)$ and the new value $f(y_m)$ preserving the fourth order convergence and f_3 is chosen such that it uses the already computed values $f(x_m)$, $f(w_m)$, $f(y_m)$ and the new value $f(z_m)$ to provide the optimal eighth order convergence.

Construction of Derivative Free Root Solver extendable to with-memory

For instance, consider the optimal fourth order two-step King's method [7],

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, m \geq 0, \\
z_m &= y_m - \frac{f(y_m)}{f'(x_m)} \frac{f(x_m) + af(y_m)}{f(x_m) + (a-2)f(y_m)}, a \in R,
\end{aligned} \tag{6}
$$

Construction of Derivative Free Root Solver extendable to with-memory

Applying the above procedure (5) we propose the following without memory optimal eighth order modification of (6) having no derivative:

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f[x_m, w_m] + qf(w_m)}, w_m = x_m + pf(x_m), m \geq 0, \\
z_m &= y_m - S(u_m)\frac{f(x_m) + af(y_m)}{f(x_m) + (a-2)f(y_m)} \\
&\quad \times \frac{f(y_m)}{f[y_m, w_m] + qf(w_m) + s(y_m - w_m)(y_m - x_m)}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{Q_m},
\end{aligned} \tag{7}
$$

where $u_m = \frac{f(y_m)}{f(x_m)}$, $a \in \mathbf{R}$, $Q_m = f[y_m, z_m] + f[z_m, y_m, x_m](z_m - y_m)$ $+f[z_m, y_m, x_m, w_m](z_m - y_m)(z_m - x_m) + t(z_m - w_m)(z_m - y_m)(z_m - x_m)$, p, q, s and t are free parameters. The following theorem holds by imposing the conditions on $S(u_m)$ to achieve optimal order of convergence for (7).

Convergence Analysis of Derivative Free Root solver

Theorem

Let g $\in I$ *be a simple root of a sufficiently differentiable function* $f : I \subseteq \mathbf{R} \to \mathbf{R}$, *where* $I \subseteq \mathbf{R}$ *is an open set and the starting point* x_0 *is close enough to* g. *Then, the scheme (7) is eighth order convergent if* $S(0) = 1, S'(0) = -1, S''(0) = -2$ *and* $S'''(0) = 0$, *and possesses the following error equations*

$$\begin{aligned} e_{m+1} &= \frac{1}{c_1^2}(c_2+q)^2(1+pc_1)^4(2aq^2pc_1^2+4aqc_2pc_1^2+2ac_2^2pc_1^2 \\ &+2c_1aq^2+2c_1qc_2+4c_1aqc_2-c_1c_3+2c_1c_2^2+2c_1c_2^2+s)\times \\ &(-t+c_1c_4+sc_2-c_2c_1c_3+2c_2^3c_1+2qc_2^2c_1+2c_1ac_2^3+4c_1qc_2^2 \\ &+2c_2c_1aq^2+2ac_2^3pc_1^2+4aqc_2^2pc_1^2+2c_2aq^2pc_1^2)e_m^8 \\ &+O(e_m^9), \end{aligned} \tag{8}$$

where, $c_k = \frac{f^{(k)}(\mathrm{g})}{k!f'(\mathrm{g})}, k \geq 2$.

It can be noted that the coefficient of e_m^8 in (8) disappears if $p = \frac{-1}{c_1}, q = -c_2, s = c_1c_3$ and $t = c_1c_4$, where $c_1 = f'(\mathrm{g})$ and $c_k = \frac{f^{(k)}(\mathrm{g})}{k!f'(\mathrm{g})}, k \geq 2$. In this way our newly suggested without memory method (7) can be extended to with-memory method by approximating the involved parameters in such a way that the local order of convergence is increased.

Construction of Four-Parametric With Memory Root Solvers

To construct with-memory method

- the free parameters p, q, s and t are calculated by the formulas $p_n = \frac{-1}{\overline{f'}(g)}, q_n = -\frac{\overline{f''}(g)}{2\overline{f'}(g)}, s_n = \frac{\overline{f'''}(g)}{6}, t_n = \frac{\overline{f^{iv}}(g)}{24}$

, for $n = 1, 2, ...,$ where $\overline{f'}, \overline{f''}(g), \overline{f'''}(g), \overline{f^{iv}}(g)$ are the best approximations to $f'(g), f''(g), f'''(g), f^{iv}(g)$,
since exact value of simple root is not known and consequently the derivatives of the function cannot be computed. The approximations $\overline{f'}, \overline{f''}(g), \overline{f'''}(g), \overline{f^{iv}}(g)$ are computed by Newton's Interpolating polynomials of appropriate degrees respectively.

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- Accordingly we replace the free parameters p, q, s and t in (7) by self-accelerators p_n, q_n, s_n, t_n and present the following with-memory root solver:

Construction of Four-Parametric With Memory Root Solvers

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f[x_m, w_m] + q_n f(w_m)}, w_m = x_m + p_n f(x_m), m \geq 2, \\
z_m &= y_m - S(u_m) \frac{f(x_m) + af(y_m)}{f(x_m) + (a-2)f(y_m)} \\
&\quad \times \frac{f(y_m)}{f[y_m, w_m] + q_n f(w_m) + s_n (y_m - w_m)(y_m - x_m)}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{Q_m},
\end{aligned} \tag{9}
$$

Construction of Four-Parametric With Memory Root Solvers

where u_m and Q_m are same as given in (7) and

$$p_n = \frac{-1}{N_4'(x_m)}, q_n = -\frac{N_5''(w_m)}{2N_5'(w_m)}, s_n = \frac{N_6'''(y_m)}{6}, t_n = \frac{N_7^{iv}(z_m)}{24}. \qquad (10)$$

The self-accelerators are calculated recursively using available information in the current and previous iterations. Hence, we use Newton's interpolation method to approximate the derivatives of f, where $N_4(x_m), N_5(w_m), N_6(y_m)$ and $N_7(z_m)$ are Newton's interpolation polynomials of degree four, five, six and seven respectively defined by:

$$\begin{aligned}
N_4(t) &= N_4(t; x_m, z_{m-1}, y_{m-1}, w_{m-1}, z_{m-2}),\\
N_5(t) &= N_5(t; w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}, z_{m-2}),\\
N_6(t) &= N_6(t; y_m, w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}, z_{m-2}),\\
N_7(t) &= N_7(t; z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}, z_{m-2}),
\end{aligned}$$

for any $m \geq 2$.

R- Order of convergence for Four-Parametric With Memory Root Solvers

Theorem

Let x_0 be an initial approximation sufficiently close to the root g *of the function $f(x)$. If the parameters p_n, q_n, s_n and t_n are recursively computed by the forms given in (10) then the convergence R-order of (9) is at least* 15.51560 *with the efficiency index* $15.51560^{\frac{1}{4}} \approx 1.98468$.

We will now prove that the with-memory method (9) has convergence order 15.51560 by applying the Herzberger's matrix method [6] provided that self-accelerators given in (10) are used.

R- Order of convergence for Four-Parametric With Memory Root Solvers

By the Herzberger's matrix method, It can be seen that the spectral radius of a matrix $A^{(s)} = (h_{ij})(1 \leq i, j \leq s)$ associated with a with-memory one-step $s-$point method $x_k = \text{f}\ (x_{k-1}, x_{k-2}, ..., x_{k-s})$ is the lower bound of the its order of convergence. The elements of this matrix are given by:

$$\begin{aligned} h_{1,j} &= \text{number of function evaluations required at point } x_{k-j}, \\ j &= 1, 2, ..., s, h_{i,i-1} = 1 \text{ for } i = 2, 3, ..., s, \\ h_{i,j} &= 0, \text{ otherwise.} \end{aligned}$$

R- Order of convergence for Four-Parametric With Memory Root Solvers

On the other hand, the spectral radius of product of the matrices $A_1, A_2, \cdots, A_s$, is the lower bound of order of an $s-$step method $\mathrm{f} = \mathrm{f}_1 \circ \mathrm{f}_2 \circ \cdots \circ \mathrm{f}_s$, where the matrices A_r correspond to the iteration steps $\mathrm{f}_r, 1 \leq r \leq s$. From the relations (9) and (10), we construct the corresponding matrices as follows:

$$
\begin{aligned}
x_{m+1} &= \mathrm{f}_1(z_m, y_m, w_m, x_m, z_{m-1}, y_{m-1}, w_{m-1}, z_{m-2}) \\
\rightarrow \quad A_1 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},
\end{aligned}
$$

Numerical Results and Comparison

For the comparison, we have taken the following test functions:

$$
\begin{aligned}
f_1(x) &= e^{x^2-3x}\sin(x)+\log(x^2+1),\ \ x_0=0.35,\ \ \mathrm{w}=0, \qquad (11)\\
f_2(x) &= e^{x^2+x\cos(x)-1}\sin(\mathrm{p}\,x)+x\log(x\sin(x)+1),\ \ x_0=0.6,\ \ \mathrm{w}=0,\\
f_3(x) &= e^{-x^2+x+2}+\sin(\mathrm{p}\,x)e^{x^2+x\cos(x)-1}+1,\ \ x_0=1.3,\ \ \mathrm{w}\approx 1.55031
\end{aligned}
$$

Numerical examples are taken from [10] to test the proposed with-memory root solver (9) in comparison with the with-memory family of methods of Kung and Traub [8] and with-memory method of Lotfi et al. [10] (1). All numerical computations are performed using the programming package Maple16 with multiple-precision arithmetic by applying 3000 fixed floating point arithmetic.

Tables 1-3 display the behavior of the approximate values for the test functions, where $A(-d)$ denotes $A \times 10^{-d}$. For all the compared with-memory methods, we have considered $p_0 = s_0 = t_0 = 0.01, q_0 = 0.1$. From the obtained results it is evident that the proposed with-memory method (9) has very fast convergence behavior than the with-memory method of Kung and Traub [8] and Lotfi et al. [10] (1).

Numerical Results and Comparison

Table 1: Results of With-memory method (1), $b_0 = q_0 = t_0 = -0.1$

Functions	$\lvert f(x_1)\rvert$	$\lvert f(x_2)\rvert$	$\lvert f(x_3)\rvert$	$\lvert f(x_4)\rvert$
$f_1(x)$	$0.2(-3)$	$0.32(-21)$	$0.11(-155)$	$0.44(-1196)$
$f_2(x)$	$0.58(-4)$	$0.24(-27)$	$0.12(-206)$	$0.98(-1592)$
$f_3(x)$	$0.45(-2)$	$0.41(-17)$	$0.10(-129)$	$0.10(-1007)$

Numerical Results and Comparison

Table 2: Results of With-Memory method (11), $g_0 = 0.1$

Functions	$\lvert f(x_1)\rvert$	$\lvert f(x_2)\rvert$	$\lvert f(x_3)\rvert$	$\lvert f(x_4)\rvert$
$f_1(x)$	$0.45(-6)$	$0.16(-51)$	$0.87(-435)$	$0.75(-3683)$
$f_2(x)$	$0.37(-1)$	$0.67(-13)$	$0.13(-110)$	$0.23(-939)$
$f_3(x)$	$0.79(-3)$	$0.16(-33)$	$0.21(-291)$	$0.34(-2477)$

Table 3: Results of With-memory method (9)

Functions	$\lvert f(x_1)\rvert$	$\lvert f(x_2)\rvert$	$\lvert f(x_3)\rvert$	$\lvert f(x_4)\rvert$
$f_1(x)$	$0.15(-6)$	$0.40(-39)$	$0.51(-391)$	$0.12(-3781)$
$f_2(x)$	$0.40(-3)$	$0.34(-20)$	$0.12(-204)$	$0.38(-2049)$
$f_3(x)$	$0.98(-5)$	$0.22(-35)$	$0.10(-365)$	$0.1(-2998)$

Dynamical Behavior

We get important information about the stability and reliability of the iterative methods by visualizing the dynamical properties of the associated iterative root solvers.We investigate the comparison of the dynamical planes associated to the with-memory root solver of Kung-Traub Method with-memroy [8] (11) to some complex functions in the complex plane using basin of attraction.

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1. The dynamical planes are obtained using technique on Matlab R2013a software as follows:
2. By taking a rectangle $[-2, 2] \times [-2, 2]$ of the complex plane, we define a mesh of 1000×1000 initial approximations.

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4. with an error approximation lower than 10^{-5} and at most 25 iteration.

Dynamical Behavior

Each initial guess is assigned with a color depending upon to the number of iterations for the iterative method to converge to any of the root of the given function.
In this technique we use colormap 'Hot'. The color of the initial point will be more intense if the sequence of the iterative method converges in less number of iterations and if it is not converging to any of the roots after maximum number of 25 iterations, then initial point is assigned with black color.

Dynamical Behavior

The proposed with-memory method (9) and Kung-Traub Method with-memroy [8] (**??**) are applied to the following complex functions:

- $p_1(z) = z^3 - 1$, with roots 1.0, $-0.5000 + 0.86605I$, $-0.5000 - 0.86605I$,

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- $p_2(z) = z^5 - 1$, with roots $1.0,\ 0.3090 + 0.95105I,\ -0.8090 + 0.58778I, -0.8090 - 0.58778I, 0.30902 - 0.95105I$

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- $p_3(z) = z^6 - \frac{1}{2}z^5 + \frac{11(i+1)}{4}z^4 - \frac{3i+19}{4}z^3 + \frac{5i+11}{4}z^2 + \frac{i-11}{4}z + \frac{3}{2} - 3i$,

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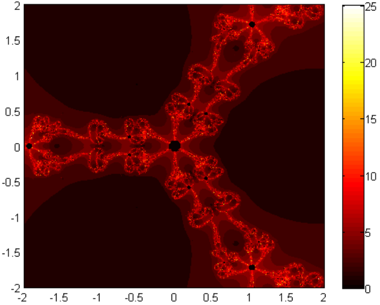
- with the solutions $-1.0068 + 2.0047i, 0.0281 + 0.9963i, 0.0279 - 1.5225i, 1.0235 - 0.9556i, 0.9557 - 0.0105i, -0.5284 - 0.5125i.$

Dynamical planes of the with-memory methods (9) and (11) for $n = 3$ and $b_0 = 0.01$ applied to the functions $p_1(z), p_2(z), p_3(z)$ are depicted in the figures 1-3. Color maps are provided and the number of iterations in which the convergence occurs. From the following figures , it is easily observed that the appearance of darker region shows that the iterative method (6) consumes less number of iterations in comparison with (11). It is concluded that the proposed with-memory method (9) is the best one because its dynamical planes has less black and dark blue regions as compared to the with-memory family of Kung and Traub (**??**) (KT).

$p_1(z) = z^3 - 1$


2
1.5
1
0.5
0
-0.5
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-1.5
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25
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15
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5
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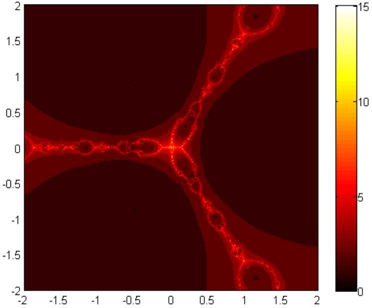
2
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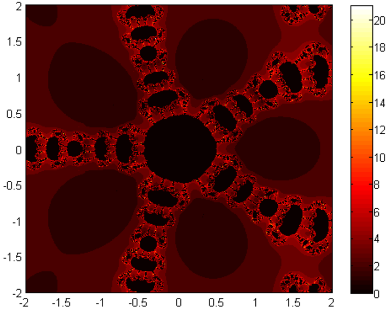
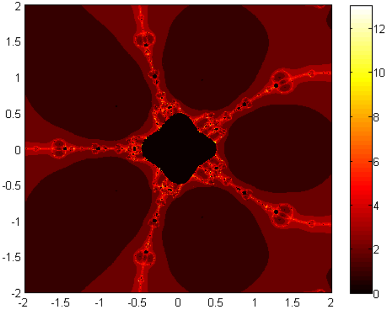
-0.5
-1
-1.5
-2
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
15
10
5
0

Color maps

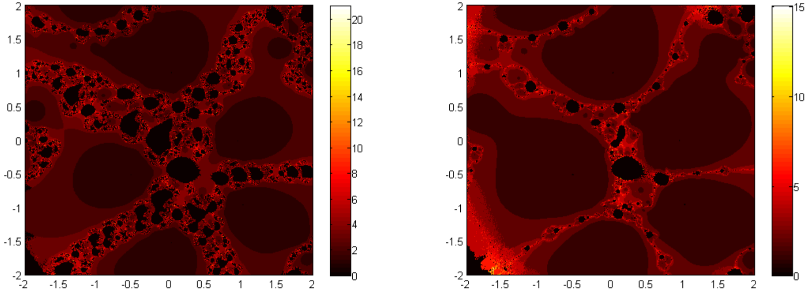
$$p_2(z) = z^5 - 1$$





Color maps

$p_2(z) = z^6 - 1$



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- Finally numerical results are presented which illustrate that the proposed with-memory iterative methods with have good enough behaviors for finding roots of nonlinear functions.

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