Oka Theory and Complex Geometry Conference

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Geometry of complex polynomial maps at infinity

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Real and Complex Jacobian conjecture

"Let $F = (F_1, \dots, F_n) : \mathbb{K}^n \to \mathbb{K}^n$, where $n \geq 1$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, be a *polynomial map*.

$JF(p)$: Jacobian matrix of F at p.

Jacobian conjecture: If

$$\det JF(p) \neq 0, \quad \forall p \in \mathbb{K}^n,$$

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- $n = 2$: The conjecture is false for in the real case! (Pinchuk, 1994)

Construction of the smallest degree (25) Pinchuk's map (Counter-example to the Real Jacobian conjecture):

Given $(x, y) \in \mathbb{R}^2$, denote

$$t = xy - 1, \quad h = t(xt + 1), \quad f = (xt + 1)^2(t^2 + y).$$

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$p = f + h$

$q = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \frac{75}{4}h^4.$

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Then the polynomial mapping $\mathcal{P} := (p, q) : \mathbb{R}^2 \to \mathbb{R}^2$ is the one such that $\det(J\mathcal{P}(x,y)) > 0, \forall (x,y) \in \mathbb{R}^2$ but $\mathcal{P}$ is not injective.

- In the two papers

• Francisco Braun, Filipe Fernandes, *Very degenerate polynomial submersions and counterexample to the real Jacobian conjecture*, Journal of Pure and Applied Algebra 227 (2023) 107345;

• Filipe Fernandes, *A new class of non-injective polynomial local diffeomorphisms on the plane*, J. Math. Anal. Appl. 507 (2022) 125736,

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It remains to investigate the real conjecture when the degree of p is 5, 6, 7 or 8.

The conjecture is still *open* for $\mathbb{K} = \mathbb{C}$ even with dimension $n = 2$.

There have been many proofs for the complex conjecture for dimension 2 but one found out mistakes in the proofs. Por example, one of newest proofs:

W. Bartenwerfer, *The Cremona problem in dimension 2*, Archiv der Mathematik. 119 (2022), 53-62

solves the conjecture positively. And the erros were found soon in:

Szymon Brazostowski and Tadeusz Krasiński, *A note on the paper "The Cremona problem in dimension 2" by Wolfgang Bartenwerfer*, arXiv:2306.03996 (6 June 2023).

An equivalent statement of the Jacobian conjecture

Let $F : \mathbb{K}^n \to \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial mapping. If $\det JF(p) \neq 0, \forall p \in \mathbb{K}^n$, then F is proper.

- The map F is proper if and only if its asymptotic set S_F is empty.

$$S_F := \{a \in \mathbb{K}^n : \exists \{\xi_k\} \subset \mathbb{K}^n, |\xi_k| \to \infty, F(\xi_k) \to a\}.$$

Structure of the talk

Part 1. *Intersection homology approach* and applications.

Part 2. *Newton polygon approach* and the complex conjecture until degree 104.

Part 3. *Asymptotic set approach* and some classes satisfying the Jacobian conjecture.

Intersection homology approach

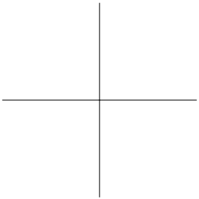
Let X be a m-dimensional semi-algebraic variety.
A semi-algebraic stratification of X is the data of a finite semi-algebraic filtration:

$$(\mathcal{S}): \quad \emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-2} \subset X^{n-1} \subset X = X^m$$

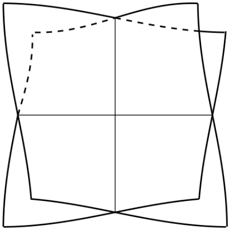
such that for every i, the set $S^i = X^i - X^{i-1}$ is either empty or a smooth manifold of pure dimension i. A connected component of S^i is called a *stratum* of X.

Example:

$\emptyset \subset \{0\} \subset$



$\subset$

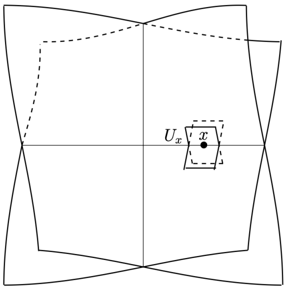


A locally topologically trivial stratification is the one such that for every $x \in X_i \setminus X_{i-1}$, $i \geq 0$, there is an open neighborhood U_x of x in X and a semi-algebraic homeomorphism

$$h : U_x \to]0,1[^i \times \mathring{c}L_x,$$

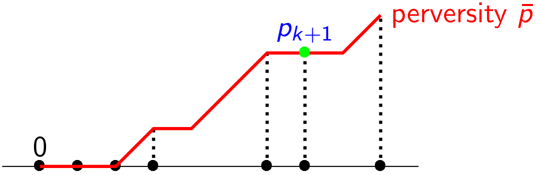
such that h maps the strata of U_x (induced stratification) onto the strata of $]0,1[^i \times \mathring{c}L_x$ (product stratification).

Example:



The definition of perversities as originally given by Goresky and MacPherson: A perversity is a $(m+1)$-tuple of integers $\bar{p} = (p_0, p_1, p_2, p_3, \ldots, p_m)$ such that

1. $p_0 = p_1 = p_2 = 0$;
2. $p_{k+1} \in \{p_k, p_k + 1\}$, for $k \geq 3$.



Example:

1. Zero perversity: $\overline{0} = (0, 0, 0, \ldots, 0)$;
2. Maximal perversity (total perversity): $\overline{t} = (0, 0, 0, 1, 2, \ldots, m-2)$.

We say that two perversities $\overline{p}$ and $\overline{q}$ are complementary if $\overline{p} + \overline{q} = \overline{t}$.

In the case of manifolds, global homological invariants like Betti numbers enjoy remarkable duality properties as stated by Poincaré (1893) and Lefschetz (1926). For smooth manifolds, the de Rham theorem (1931) and Morse theory (1934) show that it is possible to compute such topological invariants using differential forms and smooth functions. *Unfortunately, all these beautiful results fail to hold for singular varieties!* In an attempt to generalize the powerful theory of characteristic numbers to the singular case, Mark Goresky and Robert MacPherson noticed about 1973 that the failure of Poincaré duality is caused by the lack of transversal intersection of cycles on the singular locus. As a remedy, they introduced chains with well controlled intersection behaviour on the singular locus. These "intersection chains" form a complex that yields a new (co-) homology theory, called *"intersection homology"*. As a key point, the new theory yields an intersection product with suitable duality properties.

X: m-dimensional semi-algebraic variety with a locally topologically trivial stratification. A semi-algebraic subset $Y \subset X$ is $(\bar{p}, i)$- allowable if

$$\dim(Y \cap X_{m-k}) \leq i - k + p_k, \qquad \forall k.$$

Denote $IC_i^{\overline{p}}(X)$: the $\mathbb{R}$-vector subspace of $C_i(X)$ of chains ξ such that $|\xi|$ is $(\overline{p}, i)$-allowable and $|\partial\xi|$ is $(\overline{p}, i-1)$-allowable.

The i^{th} intersection homology group with perversity $\overline{p}$, denoted by $IH_i^{\overline{p}}(X)$, is the i^{th} homology group of the chain complex $IC_*^{\overline{p}}(X)$.

The intersection homology is independent of the choice of the stratification.

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The Poincaré duality holds for the intersection homology of a (singular) variety: For any orientable compact stratified semi-algebraic m-dimensional variety X, we have:

$$IH_k^{\overline{p}}(X) \simeq IH_{m-k}^{\overline{q}}(X),$$

where $\overline{p}$ and $\overline{q}$ are complementary perversities.

An intersection homology approach to the Jacobian conjecture:

Theorem (Anna Valette and Guillaume Valette, 2010).

Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping satisfying the condition of the Jacobian conjecture. Then there exist singular varieties V_F such that F is a diffeomorphism iff the intersection homology $IH_2^{\overline{p}}(V_F, \mathbb{R}) = 0$, for *any* perversity $\overline{p}$.

A. Valette and G. Valette, *Geometry of polynomial mappings at infinity via intersection homology*, Ann. I. Fourier vol. 64, fascicule 5 (2014), 2147-2163.

Idea of the construction of Valettes' varieties:

1. Consider the polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ as a real one $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$.
2. Take $\{V_i\}$ a finite covering by smooth submanifolds of $\mathbb{R}^{2n} \setminus \mathrm{Sing} F$. ($\mathrm{Sing} F$: critical points)
   Remark: F induces a diffeomorphism from V_i into its image $F(V_i)$.
3. Use **Mostowski's Seperation Lemma:** "Let U be an open, semi-algebraic subset in $\mathbb{R}^k$ and A, B two disjoint closed, semi-algebraic subsets contained in U. Then there exists a Nash function $\psi : U \to \mathbb{R}$ such that ψ is positive on A and negative on B." (A Nash function $\psi : U \to \mathbb{R}$ is an analytic one satisfying a polynomial equation $P(x, \psi(x)) = 0$).

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Thank to Mostowski's seperation technique, they can separate the images $\{F(V_i)\}$ by adding one dimension for each one to embed them in a higher dimensional space and glue them together. They get a singular variety V_F such that

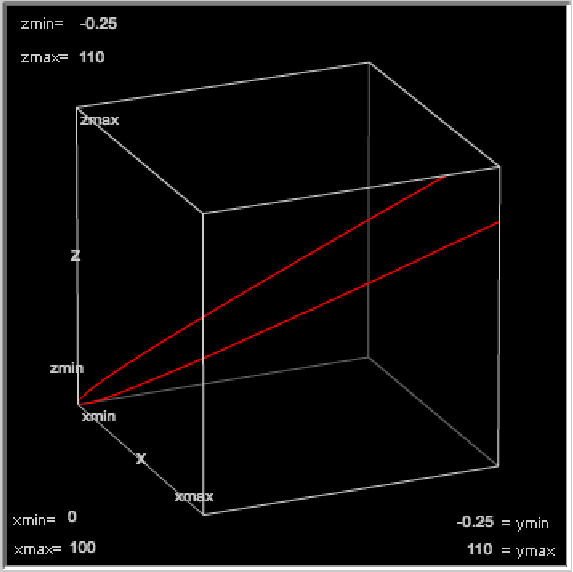
$$\mathrm{Sing} V_F \subset (S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}.$$

($K_0(F) = F(\mathrm{Sing}(F))$: critical values).

An example: Let $F : \mathbb{R} \to \mathbb{R}$ be a function defined by $F(x) = x^2$.

1. $\mathrm{Sing}(F) = \{0\}$.
2. $\mathrm{Sing}(F)$ divide $\mathbb{R}$ into two subsets:
$$U_1 = \{x \in \mathbb{R} : x > 0\} \quad , \quad U_2 = \{x \in \mathbb{R} : x < 0\}$$
3. $F(U_1) = F(U_2) = \{\alpha \in \mathbb{R} : \alpha > 0\}$.
4. Put $U := \mathbb{R} \setminus \{0\}$. There exist Nash functions:
$$\psi_1 : U \to \mathbb{R}, \quad \psi_1(x) := \frac{x}{1+x^2}$$
$$\psi_2 : U \to \mathbb{R}, \quad \psi_2(x) := \frac{-x}{1+x^2}.$$
satisfying the Mostowski's Seperation Lemma.
5. Define
$$V_F := \overline{(F, \psi_1, \psi_2)(U)} = \overline{\left\{\left(x^2, \frac{x}{1+x^2}, \frac{-x}{1+x^2}\right) : x \neq 0\right\}} \subset \mathbb{R}^3.$$

The Valette variety associated to $F : \mathbb{R} \to \mathbb{R}, \quad F(x) = x^2$:



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Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ a polynomial mapping satisfying the condition of the Jacobian conjecture. Then there exist singular varieties V_F (called *Valette varieties*) such that F is an automorphism iff its intersection homology $IH_2^{\overline{p}}(V_F, \mathbb{R}) = 0$, for *any* perversity $\overline{p}$.

Theorem (–, Anna Valette and Guillaume Valette, 2013)

The above theorem is still true for the general case $F : \mathbb{C}^n \to \mathbb{C}^n$, where $n \geq 2$, with the additional condition: "The parts at infinity of the fibres of F are complete intersections."

Thuy Nguyen, A. Valette and G. Valette, *On a singular variety associated to a polynomial mapping*, Journal of Singularities volume 7 (2013), 190-204.

Some properties of Valette varieties:

1. Lipschitz properties: There exists a semi-algebraic bi-Lipschitz mapping

$$h_F : \mathbb{R}^{2n} \setminus \mathrm{Sing}(F) \to V_F \setminus ((S_F \cup K_0(F)) \times \{0_{\mathbb{R}^p}\}.$$

2. "Good" stratification: there exists a Whitney stratification for varieties V_F.
(Whitney stratification $\Rightarrow$ locally topologically trivial stratification)

Application 1

Theorem (-, 2023)

There exists a Valette variety $V_{\mathcal{P}}$ associated to the Pinchuk map $\mathcal{P}$ such that

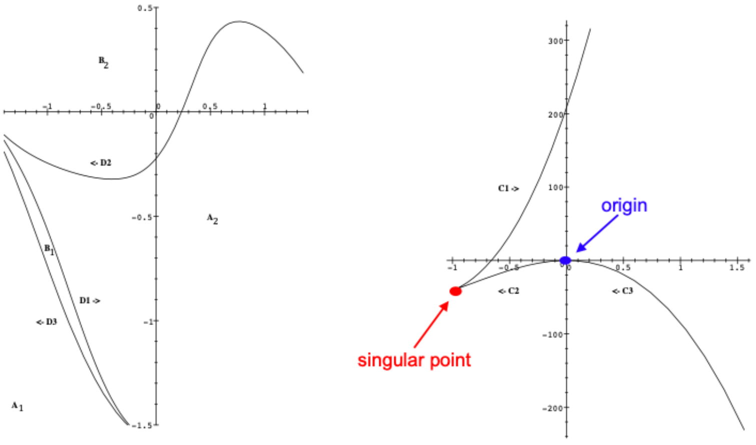
$$IH_1^{\bar{0},c}(V_{\mathcal{P}}) = IH_1^{\bar{0},\mathrm{cl}}(V_{\mathcal{P}}) = 0.$$

Remark

The "*real version*" of Valettes' result *is no longer true!*

Thuy Nguyen, *A singular variety associated to the smallest degree Pinchuk map*, Matemática Contemporânea, Sociedade Brasileira de Matemática, 2023.

Idea of the proof: Use the behaviours of the asymptotic variety of the Pinchuk's map described by L. A. Campbell, (1996-2001).



The asymptotic flower

The asymptotic variety

Application 2:

Theorem (- , Maria Aparecida Soares Ruas, 2018)

Let $G = (G_1, \ldots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 2$, be a generic polynomial mapping such that $K_0(G) = \emptyset$ (local submersion) and the parts at infinity of the fibres of G are complete intersections. Then there exist singular varieties $\mathcal{V}_G$ associated to G such that if $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R})$ is trivial then the bifurcation set $B(G)$ is empty (here $\bar{t}$ is the total perversity).

Thuy Nguyen and Ruas, M.A.S., *On singular varieties associated to a polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, Asian Journal of Mathematics, v.22, p.1157-1172, 2018.

Idea of the proof:

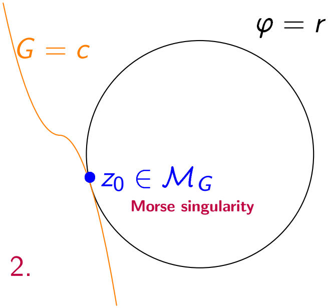
- $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, $\quad K_0(G) = \emptyset$, $\quad G : \mathbb{R}^{2n} \to R^{2n-2}$.
- $\varphi : \mathbb{C}^n \to \mathbb{R}$, $\quad \varphi(z) = \sum_{i=1}^{n} a_i |z_i|^2$

($\varphi = r$: $2n$-dimensional real sphere).

- $\mathcal{M}_G = \mathrm{Sing}(G, \varphi)$.

(G, φ): mixed function

(M. Oka, *A Non-degenerate mixed function*, Kodai Math, 2010)



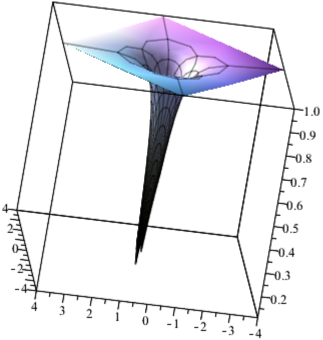
G: generic

$\Rightarrow \dim_{\mathbb{R}} \mathcal{M}_G = 2n - 2$.

Broughton's Example:

$$G : \mathbb{C}^2 \to \mathbb{C}, \qquad G(z,w) = z + z^2 w,$$

$$K_0(G) = \emptyset, \quad B(G) = \{0\}$$



Theorem (Hà Huy Vui and Nguyễn Tất Thắng, 2011).

Let $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 2$, be a *local submersion* polynomial mapping. If the parts at infinity of the fibres of G are complete intersections, then the bifurcation set $B(G)$ of G is empty if and only if the Euler characteristic of $G^{-1}(t)$ is a constant for any $t \in \mathbb{C}^{n-1}$.

V. H. Ha, T. T. Nguyen *On the topology of polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$*, International Journal of Mathematics, Vol. 22, No. 3 (2011) 435-448.

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H. Ha, *Nombres de Łojasiewicz et singularités à l'infini des polynômes de deux variables complexes*, C. R. Acad. Sci. Paris Ser. I **311** (1990) 429-432.

V. H. Ha and L.D. Tráng, *Sur la topologie des polynomes complexes*, Acta Math. Vietnamica **9** (1984) 21-32.

M. Suzuki, *Propriétés topologiques des polynomes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$*, J. Math. Soc. Japan **26** (1974) 241-257.

Comparison with Hà Huy Vui's and Nguyễn Tất Thắng's Result

V.H.Ha and T.T. Nguyen Resuls

1. $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$
2. $K_0(G) = \emptyset$
3. $\dim_{\mathbb{C}}\{\hat{G}_i(z)\}_{i=1,\dots,n} = 1$
   $\hat{G}_i$: the leading form of G_i
4. $\chi(G^{-1}(t)) = \text{const.} \Leftrightarrow B(G) = \emptyset.$

Our Result

1. $G : \mathbb{C}^n \to \mathbb{C}^{n-1}$
2. $K_0(G) = \emptyset$
3. $\mathrm{Rank}_{\mathbb{C}}(D\hat{G}_i)_{i=1,\dots,n-1} > n-3,$
   $\hat{G}_i$: the leading form of G_i
4. $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) = 0 \Rightarrow B(G) = \emptyset.$

A consequence: *Relation between the Euler characteristic and Intersection Homology*

Corollary (-, Maria Aparecida Soares Ruas, 2018)

Let $G = (G_1, \ldots, G_{n-1}) : \mathbb{C}^n \to \mathbb{C}^{n-1}$, where $n \geq 2$, be a generic polynomial mapping such that $K_0(G) = \emptyset$ and the parts at infinity of the fibres of G are complete intersections. If there exist $t_1, t_2 \in \mathbb{C}^{n-1}$ such that $\chi(G^{-1}(t_1)) \neq \chi(G^{-1}(t_1))$, then there is singular variety $\mathcal{V}_G$ associated to G such that $IH_2^{\bar{t}}(\mathcal{V}_G, \mathbb{R}) \neq 0$.

Newton polygon approach:

The complex conjecture until degree 104

Newton polygon

Definition

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial of two variables x and y, *i.e.*, f can be written as a finite sum:

$$f = \sum_{i,j} a_{ij} x^i y^j.$$

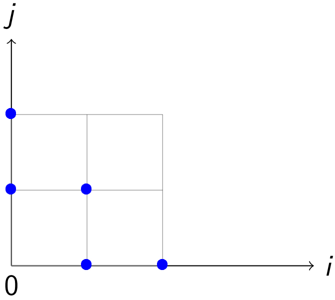
The *support* of f, denoted by $\mathrm{Supp} f$, is defined by:

$$\mathrm{Supp} f = \{(i,j) : a_{ij} \neq 0\}.$$

The Newton polygon of f, denoted by $N(f)$, is the convex hull of the origin and $\mathrm{Supp} f$:

$$N(f) = \mathrm{Conv}\left(\{(0,0)\} \cup \{(i,j) : a_{ij} \neq 0\}\right).$$

Example 1: $f(x,y) = x + x^2 + xy + y + y^2$.



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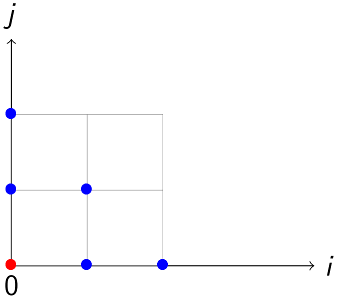

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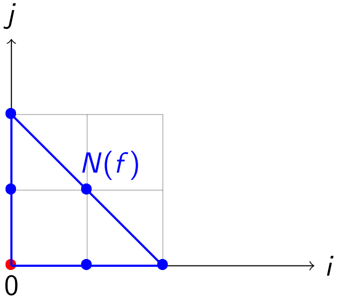

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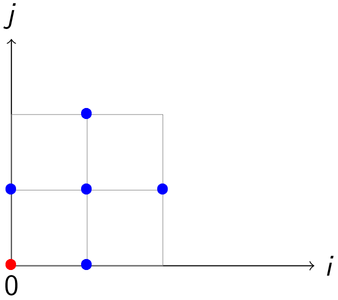

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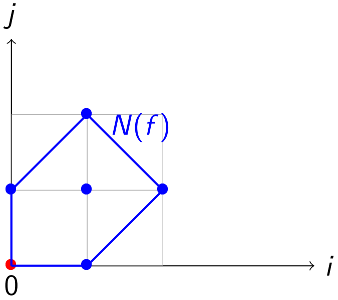

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S. S. Abhyankar, *Expansion Techniques in Algebraic Geometry*, Tata Institute of Fundamental Research, 1977
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Principal results by Newton polygon approach

Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping.

1. **(Abhyankar, 1977).** If $\deg(f)$ divide $\deg(g)$ or $\deg(g)$ divide $\deg(f)$ then the conjecture is true.
2. **(Appelgate, Onishi, Nagata, 1985).** Let $d := \gcd(\deg(f), \deg(g))$, the greatest common divisor of degrees of f and g. Then If $d \leq 8$ or d is a prime number then the conjecture is true.
3. **(Nagata, 1985).** If $\deg(f)$ or $\deg(g)$ is a product of at most two prime numbers, then the conjecture is true.

With a highly non-trivial proof, Moh proved:

Theorem (Moh, 1983)

Let $F = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial mapping. The 2-dimensional complex Jacobian conjecture is true for $\deg(F) \leq 100$.

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The 2-dimensional complex Jacobian conjecture is true for $\deg(F) \leq 104$.

Idea of Proof: Using the principal results obtained by Newton polygon approach and the following result of Żołądek obtained by the so-called Newton-Puiseux chart

Theorem (Żołądek, 2008)

The Jacobian conjecture satisfies for maps with $\gcd(\deg(f), \deg(g)) \leq 16$ and for maps with $\gcd(\deg(f), \deg(g))$ equals to 2 times a prime.

For $\deg(F) = 105$, we have:

Proposition

If the conjecture satisfies for the cases where $(\deg(f), \deg(g))$ is

$$(42, 105), \quad (63, 105), \quad (70, 105), \quad (84, 105)$$

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A work increasing the degree of Moh's result:
Jorge A. Guccione, Juan J. Guccione, Rodrigo Horruitiner and Christian Valqui, *Increasing the degree of a possible counterexample to the Jacobian conjecture from 100 to 108*, arXiv:2204.14178, April 2022.
(the authors combinie previous complicated techniques, they list all the pairs (f, g) with degree less than 125 for any hypothetical counterexample to the plane Jacobian conjecture and discard them all)

Asymptotic approach:

Some classes satisfying the conjecture

An equivalent statement of the Jacobian conjecture

Let $F : \mathbb{K}^n \to \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a polynomial mapping. If $\det JF(p) \neq 0, \forall p \in \mathbb{K}^n$, then F is proper.

- The map F is proper if and only if its asymptotic set S_F is empty.

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Motivation: *Study a subset (sufficient large) of non-proper mappings and investigate the Condition of the Jacobian conjecture for this subset.*

This work is inspired by Pinchuk's counter-examples.

Pinchuk's map (Counter-example to the real Jacobian conjecture):

Given $(x, y) \in \mathbb{R}^2$, let:

$t = xy - 1, \quad h = t(xt + 1), \quad f = (xt + 1)^2(t^2 + y).$

Considere

$p = f + h$

$q = -t^2 - 6th(h + 1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75fh^3 - \frac{75}{4}h^4.$

Then (p, q) satisfies the Jacobian condition but it is not injective.

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Observation:

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2. The sequence $\{z_k\} = \{(k, 1/k)\} \subset \mathbb{R}^2$ tends to infinity but $t(z_k) = h(z_k) = 0$ and $f(z_k) = 1/k$. Then the Pinchuk maps (p, q), being polynomial maps of new variables t, h and f, satisfies that $(p(z_k), q(z_k))$ does not tend to infinity, in other words (p, q) is non-proper. Then (p, q) is not an automorphism.

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We define such variables t, h, f so-called *non-proper variables*.

Consider a polynomial mapping $F : \mathbb{K}^2 \to \mathbb{K}^2$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

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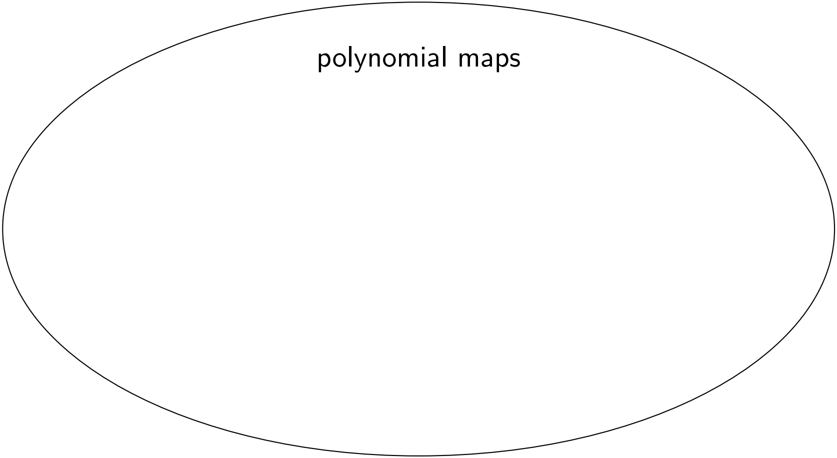
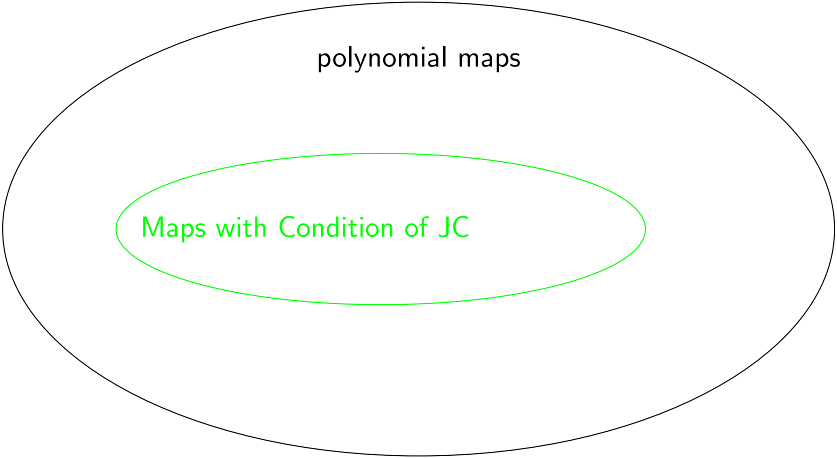
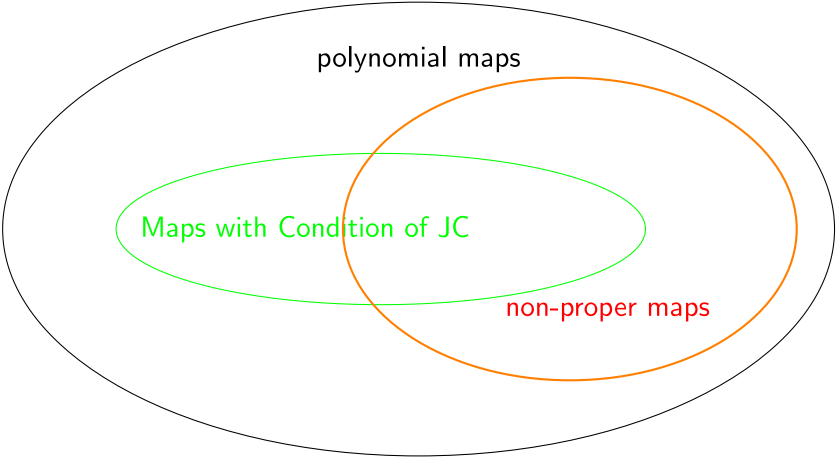
   Remark: Every mapping written as a polynomial mapping of non-proper variables is non-proper and vice-versa.
2. Classify the non-proper mappings into two types:
   - the first one does not satisfy the condition of the Jacobian conjecture (then we obtain a class of polynomial mapping satisfying the Jacobian conjecture);
   - otherwise a non-proper mapping satisfying the Jacobian condition (if there exists) is a one of the second type.


polynomial maps


polynomial maps
Maps with Condition of JC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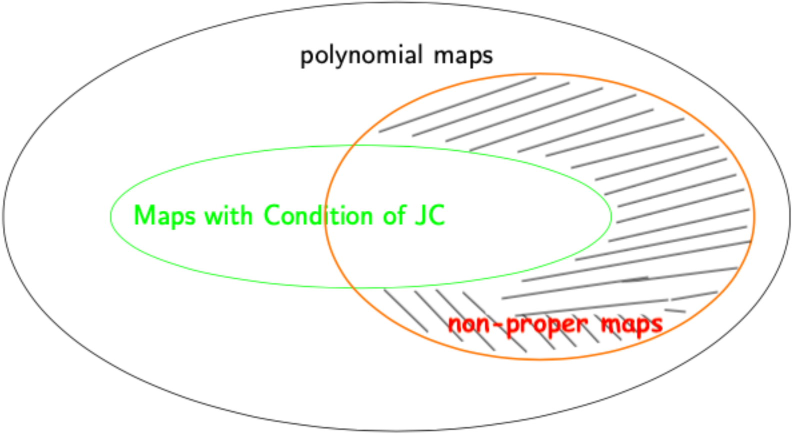
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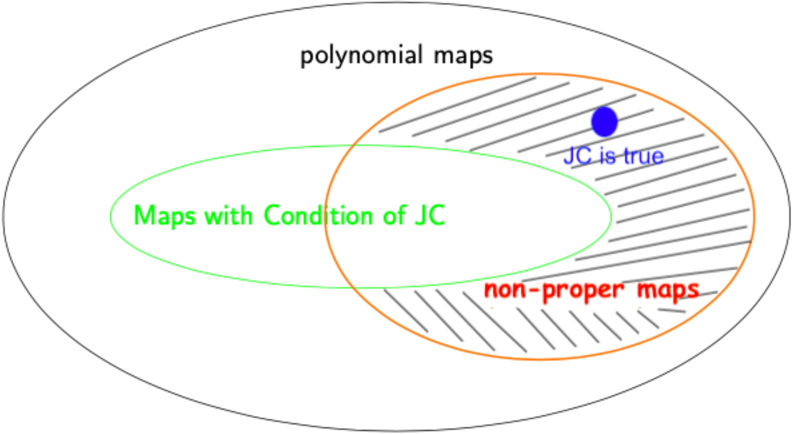
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JC is true
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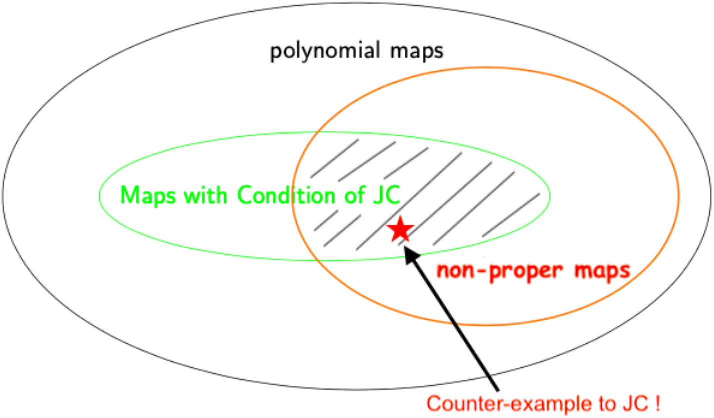
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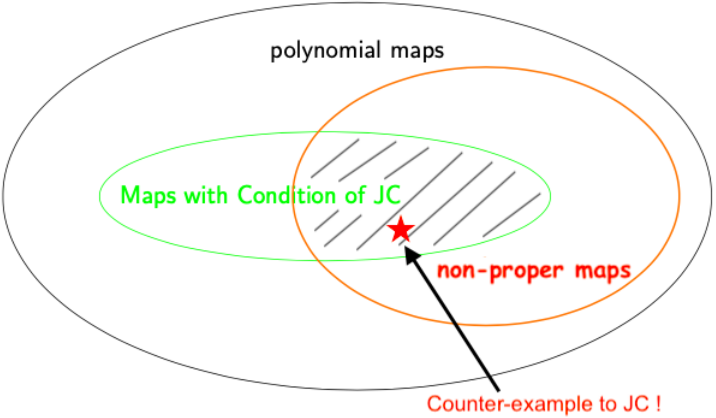
Counter-example to JC !



Pinchuk maps (counter-examples to the real Jacobian conjecture) are non-proper maps satisfying the condition of the Jacobian conjecture.

Non-proper variables:

Definition

Let $u : \mathbb{K}^2 \to \mathbb{K}$ be a polynomial function. We say that the function u is a *non-proper variable* if there is a sequence $\{(x_k, y_k)\}_{k\in\mathbb{N}} \subset \mathbb{K}^2$ such that $\lim_{k\to\infty} \|(x_k, y_k)\| = \infty$ and $\lim_{k\to\infty} u(x_k, y_k) = c \in \mathbb{K}$.

Classify non-proper variables:

Definition

Let $u_0, u_1, \ldots, u_n : \mathbb{K}^2 \to \mathbb{K}$ be non-proper variables. We say that they are *dependent non-proper variables* if there exists $z \in \mathbb{K}^2$ such that the two vectors

$$\left(\frac{\partial u_0}{\partial x}(z), \frac{\partial u_1}{\partial x}(z), \ldots, \frac{\partial u_n}{\partial x}(z)\right),$$

$$\left(\frac{\partial u_0}{\partial y}(z), \frac{\partial u_1}{\partial y}(z), \ldots, \frac{\partial u_n}{\partial y}(z)\right)$$

are *linearly dependent* in $\mathbb{K}^{n+1}$. Otherwise, the non-proper variables $u_0, u_1, \ldots, u_n$ are *independent non-proper variables*.

Theorem: *If $u_0, u_1, \ldots, u_n$ are dependent then any polynomial mapping $F(u_0, u_1, \ldots, u_n)$ does not satisfy the condition of the Jacobian conjecture.*

Theorem: Consider the set of non-proper variables from $\mathbb{K}^2$ to $\mathbb{K}$:

$$u_0 = y, u_1 = x - bx^r y^s, \ldots, u_n = x - bx^{nr} y^{ns}, \quad (b \in \mathbb{K},\ r, s > 0)$$

Then the variables $u_0, u_1, \ldots, u_n$ are non-proper and

1. If $r = 1$, then the non-proper variables $u_0, u_1, \ldots, u_n$ are dependent.
2. If $r > 1$, then for a fixed natural number $n \geq 2$, the non-proper variables $u_0, u_1, \ldots, u_n$ are independent.

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A remark: Denote by $\mathcal{C} := \bigcup_{r=1}^{\infty} \mathcal{C}_r$.

- $\mathcal{C}_1$ is a class where Jacobian conjectrue is true;
- If there exists a polynomial mapping belonging to some class $\mathcal{C}_r$, $r \geq 2$ and satisfy the Jacobian condition, then it is a counter-example to the conjecture.

A class of polynomial mapping satisfying the Jacobian conjecture:

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Motivation for studying this class: Newton polygon contains two points (1, 0) and (0, 1) (for the complex case, a mapping $(f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying the Jacobian condition when the Newton polygons $N(f)$ and $N(g)$ contain 2 points (1, 0) and (0, 1).

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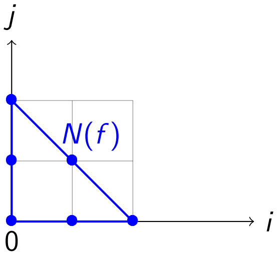

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Tusen Takk!!!!