A Family of Optimal Fourth Order Weighted Mean Based Methods For Solving System of Nonlinear Equations

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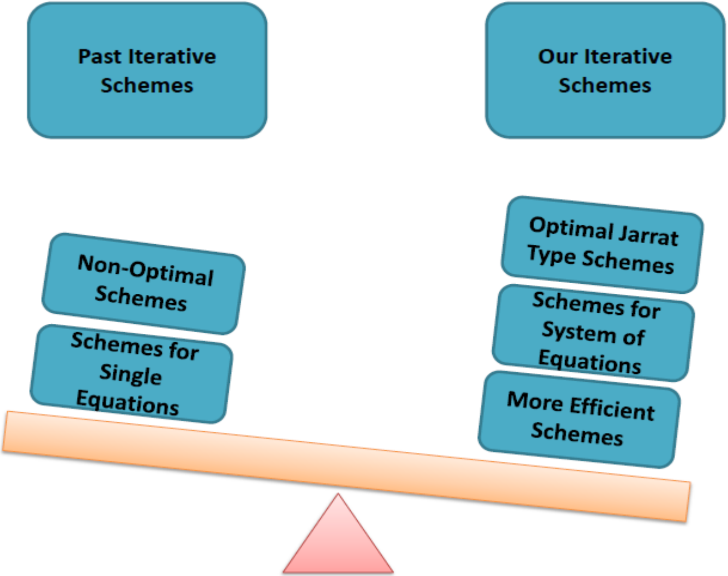
Motivation



Past Iterative Schemes
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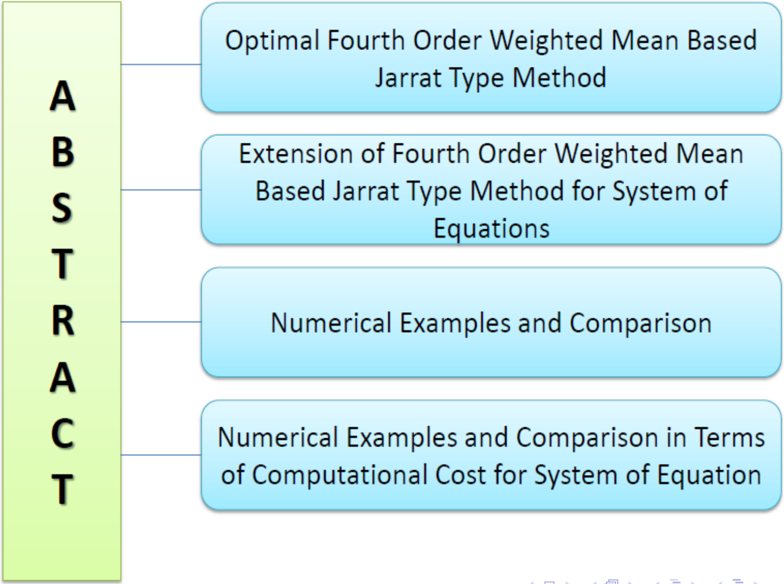
Abstarct


A B S T R A C T
Optimal Fourth Order Weighted Mean Based Jarrat Type Method
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Newton's Theorem

Newton's theorem(1) is in main focus of researchers. A breif description that how the integral in (1)is approximated by different researchers is given here:

$$f(x) = f(x_i) + \int_{x_i}^{x} f'(t)dt. \tag{1}$$

Method of Weerakon and Fernando

Weerakon and Fernando[14] used *trapezoidal rule* to approximate the integral in (1) and got the following two step third order scheme:

$$
\begin{aligned}
z_{i+1} &= z_i - \frac{f(z_i)}{f'(z_i)}, \\
x_{i+1} &= x_i - \frac{2f(x_i)}{f'(x_i) + f'(z_{i+1})}.
\end{aligned} \tag{2}
$$

The above method (2) is also called trapezoidal or arithmetic mean Newton's method. It is notable that scheme(2) having third order convergence is not optimal as conjectured by Kung and Traub.

Method of Ozban and Homeier

If *harmonic mean* is used to approximate the integral in (1), we attain with a two point third order convergent method of Ozban [11] as follows:

$$x_{i+1} = x_i - \frac{f(x_i)(f'(x_i) + f'(z_{i+1}))}{2f'(x_i)f'(z_{i+1})}, \quad (3)$$

$$\text{where, } z_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)}.$$

Homeier[6], also developed a third order convergent scheme as follows:

$$x_{i+1} = x_i - \frac{f(x_i)}{2}\left(\frac{1}{f'(x_i)} + \frac{1}{f'(y_i)}\right). \quad (4)$$

Jarrat Type Methods

Iterative scheme which uses *two derivative evaluation* and *one functional evaluation* with *fourth* order of convergence is known as *Jarrat type schemes* [7], which is given by:

$$\begin{aligned} x_{i+1} &= x_i - \left[1 - \frac{2}{3}\frac{f'(y_i) - f'(x_i)}{3f'(y_i) - f'(x_i)}\right]\left[\frac{f(x_i)}{f'(x_i)}\right], \\ \text{where, } y_i &= x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}. \end{aligned} \tag{5}$$

Khattri's Jarrat Type Method

In [10], Khattari has developed the following fourth order scheme which requires two evaluations of the derivative and one functional evaluation:

$$
\begin{aligned}
y_i &= x_i - \frac{2}{3}\frac{f(x_i)}{f'(x_i)}, \\
x_{i+1} &= x_i - \frac{f(x_i)}{f'(x_i)}\left[1 + \sum_{j=1}^{4}\beta_j\left(\frac{f'(y_i)}{f'(x_i)}\right)^j\right], \quad \text{where } \beta_j \in R.
\end{aligned} \tag{6}
$$

Chun's Jarrat Type Method

Chun [3], added his contribution by making another fourth order scheme given by:

$$\begin{aligned} y_i &= x_i - \frac{4}{3}\frac{f(x_i)}{f'(x_i)}, \quad \text{where } z_i = \frac{x_i + y_i}{2}, \\ x_{i+1} &= x_i - \frac{f(x_i)}{f'(z_i)}\left[1 + \frac{1}{4}\frac{f'(z_i) - f'(x_i)}{f'(x_i)} + \frac{3}{8}\left(\frac{f'(z_i) - f'(x_i)}{f'(x_i)}\right)^2\right] \end{aligned} \tag{7}$$

Optimal fourth Order Weighted Mean Based Method

In this contribution, we suggest a new optimal fourth order Jarratt type scheme and then provide its extension for solving the systems of nonlinear equations. We consider two step scheme in which first step is same as in Jarratt's scheme and in second step we use weight functions to achieve optimal convergence given as:

$$\begin{aligned} y_i &= x_i - \beta \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} &= x_i - \left[S\left(\mathcal{F}_i\right) \times T\left(\mathcal{L}_i\right)\right] \frac{f\left(x_i\right)}{f'\left(z_i\right)}, \text{ where } \ z_i = \frac{x_i + y_i}{2}, \end{aligned} \tag{8}$$

where $\mathcal{L}_i = \frac{f(x_i)}{f'(x_i)}$, $\mathcal{F}_i = \frac{f'(z_i)}{f'(x_i)}$ and $T\left(\mathcal{L}_i\right)$ and $S\left(\mathcal{F}_i\right)$ represents real valued weight functions chosen such that new scheme (8) achieves optimal fourth order convergence.

Order Of Convergence

Theorem

Let $\omega \in D$ be a simple zero of sufficiently smooth function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in the neighborhood of the root in interval I. Then for $\beta = \frac{4}{3}$, the new without memory scheme (8) achieves optimal order 4 under following conditions on weights

$$\begin{aligned} S(1) &= 1, \quad S^{'}(1) = \frac{1}{4}, \quad S^{''}(1) = \frac{3}{4}, \quad \left|S^{(3)}(1)\right| < +\infty, \\ T(0) &= 1, \quad T^{'}(0) = T^{''}(0) = 0, \quad \left|T^{(3)}(0)\right| < +\infty, \end{aligned} \tag{9}$$

and satisfies the error equation given by

$$e_{i+1} = (\frac{7}{3}c_2^3 - c_2c_3 + \frac{1}{9}c_4 + \frac{32}{81}S^{(3)}(1)c_2^3 - \frac{1}{6}T^{(3)}(0))e_i^4 + O(e_i^5). \tag{10}$$

Extension of New Family for Multivariate Case

In this section we extend our new family of optimal Jarratt type scheme for solving systems of nonlinear equations. We give a special case for our new family given below by defining weight functions satisfying the conditions of the above theorem as follows

$$
\begin{aligned}
S(\mathcal{F}_i) &= 1+\frac{1}{4}\left(\frac{f^{'}(z_i)}{f^{'}(x_i)}-1\right)+\frac{3}{8}\left(\frac{f^{'}(z_i)}{f^{'}(x_i)}-1\right)^2, \qquad (11)\\
T(\mathcal{L}_i) &= 1.
\end{aligned}
$$

Thus, we obtain a new Jarratt type fourth order method as:

$$
\begin{aligned}
y_i &= x_i-\frac{4}{3}\frac{f(x_i)}{f'(x_i)} \qquad (12)\\
x_{i+1} &= x_i-\left[\left(1+\frac{1}{4}\left(\frac{f^{'}(z_i)}{f^{'}(x_i)}-1\right)+\frac{3}{8}\left(\frac{f^{'}(z_i)}{f^{'}(x_i)}-1\right)^2\right)\right]\frac{f(x_i)}{f^{'}(z_i)},
\end{aligned}
$$

Extension of New Family for Multivariate Case

where $z_i = \frac{x_i+y_i}{2}$. Now, we extend our iterative method to multivariate case as:

$$\mathbf{X}^{(i+1)} = \mathbf{X}^{(i)} - \mathbf{F}^{'}(\mathbf{Z}^{(i)})^{-1}\mathbf{F}(\mathbf{X}^{(i)}) \times [(\mathbf{I} + \frac{1}{4}(\mathcal{F}(\mathbf{X}^{(i)}) - \mathbf{I}) + \frac{3}{8}(\mathcal{F}(\mathbf{X}^{(i)}) - \mathbf{I})^2)], \quad (13)$$

where $\mathbf{Y}^{(i)}(\mathbf{X}) = \mathbf{X}^{(i)} - (\frac{4}{3})[\mathbf{F}^{'}(\mathbf{X}^{(i)})]^{-1}\mathbf{F}(\mathbf{X}^{(i)})$,
$\mathcal{F}(\mathbf{X}^{(i)}) = [\mathbf{F}^{'}(\mathbf{X}^{(i)})]^{-1}\mathbf{F}^{'}(\mathbf{Y}^{(i)})$, $\mathbf{I}$ is the $\mathbf{n} \times \mathbf{n}$ identity matrix and
$\mathbf{F}(\mathbf{X}^{(i)}) = (f_1(x_1), f_2(x_2), ..., f_n(x_2))$.

Extension of New Family for Multivariate Case

Let $\mathbf{F} : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be sufficiently Fréchet differentiable in a convex set $A_0 \subset A$, where A_0 is an open convex neighborhood of $\mathbf{X}_0$, $\mathbf{X}^{(i)}$ be approximate root of the exact root ω and $\mathbf{E}^{(i)} = \mathbf{X}^{(i)} - \omega$, we can write the ith derivative of $\mathbf{F}$ at $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{h} \geq 1$,is $i-$linear function
$\mathbf{F}^{(i)}(\mathbf{h}) : \mathbb{R}^{(n)} \times \mathbb{R}^{(n)} \times ... \times \mathbb{R}^{(n)} \longrightarrow \mathbb{R}^{(m)}$ such that
$\mathbf{F}^{(i)}(\mathbf{x}^{(i)})\mathbf{h}i \in \mathbb{R}^m$described in [2].

The first derivative involving in the Taylor's series can be written as

$$[\mathbf{F}'(\mathbf{x}^{(i)})\mathbf{h}\mathbf{k}]^T = \mathbf{k}^T\mathbf{S}(\mathbf{x}^{(i)})\mathbf{h} \tag{14}$$

and second derivative may be

$$[\mathbf{F}''(\mathbf{x}^{(i)})\mathbf{h}\mathbf{k}]^T = (\mathbf{k}^T\mathbf{S}_1(\mathbf{x}^{(k)}\mathbf{h}, \mathbf{k}^T\mathbf{S}_2(\mathbf{x}^{(i)}\mathbf{h}), \ldots\mathbf{k}^T\mathbf{S}_m(\mathbf{x}^{(i)}\mathbf{h})), \tag{15}$$

where $S_1(\mathbf{x}), \ldots, S_m(\mathbf{x})$ are the Hessian matrices of $f_1, \ldots, f_m$ at x. Therefore Taylor's series for $n-$dimension case can be written as

$$\mathbf{F}(\mathbf{X}^{(i)}) = \mathbf{F}'(\boldsymbol{\omega})\left[\mathbf{E}^{(i)} + \mathbf{C}_2(\mathbf{E}^{(i)})^2 + \mathbf{C}_3(\mathbf{E}^{(i)})^3 + \mathbf{C}_4(\mathbf{E}^{(i)})^4\right] + O((\mathbf{E}^{(i)})^5) \tag{16}$$

where $\mathbf{C}_i = \frac{1}{i!}[\mathbf{F}'(\mathbf{X}_0)]^{-1}\mathbf{F}^{(i)}(\mathbf{X}_0)$, $\mathbf{F}'(\mathbf{X}_0)$ is continuous and nonsingular and $\mathbf{X}^{(0)}$ is closer to ω. By using above Taylor's expansion we may prove the next theorem.

Convergence Analysis

Theorem

Let A_0 be a convex set containing the root ω of $\mathbf{F}(\mathbf{X}) = \mathbf{0}$ and $\mathbf{F} : A \subseteq \mathbb{R}^n \to \mathbb{R}^n$, be four-time Fréchet differentiable in A_0 which has the Jacobian matrix continuous and non-singular in A. Then the new method 13 has order of convergence four satisfying the following error equation.

$$\mathbf{X}^{(i+1)} = \left(\frac{7}{3}c_2^3 - c_2 c_3 + \frac{1}{9}c_4\right)(\mathbf{E}^{(i)})^4 + O(\mathbf{E}^{(i)})^5 \tag{17}$$

Numerical Results For Univariate Case

Table 1: Test Functions and Their Roots

Numerical Example	**Exact Zero**
$f_1(x) = \sin 3x + x\cos x$	$\omega = 0.0000000000000000$
$f_2(x) = x + \cos x^2 - \frac{1}{2}$	$\omega = -0.4747149936699329$
$f_3(x) = (x+2)e^x - 1$	$\omega = -0.4428544010023885$
$f_4(x) = \sin^{-1}(x^2 - 1) - \frac{1}{2}x + 1$	$\omega = 0.5948109683983691$
$f_5(x) = x^2 e^x - \sin x$	$\omega = 0.0000000000000000$

Numerical Results For Univariate Case

Table 2: Comparison of Various Iterative Methods for $f_1(x)$

$f_1(x)$ $x_0 = 0.4$	NM	KM	CM	SM	SMNF1
n	9	D	D	D	7
$\lvert x_1 - \omega \rvert$	$3.02\ 10^{-1}$	$2.56\ 10^{0}$	$7.05\ 10^{-1}$	$6.07\ 10^{-1}$	$3.80\ 10^{-1}$
$\lvert f_1(x_1) \rvert$	$1.075\ 10^{0}$	$1.16\ 10^{0}$	$1.39\ 10^{0}$	$1.46\ 10^{0}$	$1.26\ 10^{0}$
$\lvert x_2 - \omega \rvert$	$9.4\ 10^{-2}$	$3.0\ 10^{0}$	$2.10\ 10^{0}$	$35854.07\ 10^{0}$	$1.43\ 10^{-3}$
$\lvert f_1(x_2) \rvert$	$3.72\ 10^{-1}$	$2.64\ 10^{0}$	$1.03\ 10^{0}$	$21526.38\ 10^{0}$	$5.69\ 10^{-3}$
$\lvert x_3 - \omega \rvert$	$2.13\ 10^{-3}$	$2.11\ 10^{0}$	$95.83\ 10^{0}$	35853.13	$4.12\ 10^{-12}$
$\lvert f_1(x_3) \rvert$	$8.54\ 10^{-3}$	$1.03\ 10^{0}$	$2.23\ 10^{0}$	$10417.75\ 10^{0}$	$1.64\ 10^{-11}$
$\lvert x_4 - \omega \rvert$	$2.43\ 10^{-8}$	$883.95\ 10^{0}$	$95.80\ 10^{0}$	$35853.41\ 10^{0}$	$2.88\ 10^{-46}$
$\lvert f_1(x_4) \rvert$	$9.74\ 10^{-8}$	$344.38\ 10^{0}$	$8.81\ 10^{-8}$	$290.79\ 10^{0}$	$1.15\ 10^{-45}$
$\lvert x_5 - \omega \rvert$	$3.61\ 10^{-23}$	$884.35\ 10^{0}$	$95.80\ 10^{0}$	$35853.42\ 10^{0}$	$6.91\ 10^{-183}$
$\lvert f_1(x_5) \rvert$	$1.44\ 10^{-22}$	$1.11\ 10^{0}$	$3.90\ 10^{-157}$	$1.55\ 10^{-4}$	$2.76\ 10^{-182}$
$\lvert x_6 - \omega \rvert$	$1.17\ 10^{-67}$	$884.35\ 10^{0}$	$95.80\ 10^{0}$	$35853.42\ 10^{0}$	$2.28\ 10^{-729}$
$\lvert f_1(x_6) \rvert$	$4.71\ 10^{-67}$	$5.21\ 10^{-13}$	$4.56\ 10^{-635}$	$1.25\ 10^{-29}$	$9.15\ 10^{-729}$

Numerical Results For Univariate Case

Table 3: Comparison of Various Iterative Methods for $f_2(x)$

$f_2(x)$ $x_0 = -0.3$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	10	5	5	6	5
$\lvert x_1 - \omega \rvert$	$1.12\ 10^{-2}$	$3.64\ 10^{-3}$	$8.67\ 10^{-5}$	$1.13\ 10^{-3}$	$1.43\ 10^{-4}$
$\lvert f_2(x_1) \rvert$	$1.36\ 10^{-2}$	$4.41\ 10^{-3}$	$1.05\ 10^{-4}$	$1.37\ 10^{-3}$	$1.28\ 10^{-3}$
$\lvert x_2 - \omega \rvert$	$6.99\ 10^{-5}$	$2.06\ 10^{-14}$	$2.53\ 10^{-17}$	$2.0\ 10^{-12}$	$1.26\ 10^{-12}$
$\lvert f_2(x_2) \rvert$	$8.47\ 10^{-5}$	$2.5\ 10^{-14}$	$3.07\ 10^{-17}$	$2.43\ 10^{-12}$	$1.52\ 10^{-12}$
$\lvert x_3 - \omega \rvert$	$2.67\ 10^{-9}$	$2.12\ 10^{-55}$	$1.84\ 10^{-67}$	$1.98\ 10^{-47}$	$2.55\ 10^{-48}$
$\lvert f_2(x_3) \rvert$	$3.24\ 10^{-9}$	$2.58\ 10^{-55}$	$2.23\ 10^{-67}$	$2.40\ 10^{-47}$	$3.09\ 10^{-48}$
$\lvert x_4 - \omega \rvert$	$3.91\ 10^{-18}$	$2.41\ 10^{-219}$	$5.20\ 10^{-268}$	$1.91\ 10^{-187}$	$4.30\ 10^{-191}$
$\lvert f_2(x_4) \rvert$	$4.74\ 10^{-18}$	$2.92\ 10^{-219}$	$6.31\ 10^{-268}$	$2.32\ 10^{-187}$	$5.21\ 10^{-191}$
$\lvert x_5 - \omega \rvert$	$8.36\ 10^{-36}$	$3.97\ 10^{-875}$	0	$1.65\ 10^{-747}$	0
$\lvert f_2(x_5) \rvert$	$1.01\ 10^{-35}$	$4.82\ 10^{-875}$	0	$2.00\ 10^{-747}$	0
$\lvert x_6 - \omega \rvert$	$3.82\ 10^{-71}$	0	0	0	0
$\lvert f_2(x_6) \rvert$	$4.63\ 10^{-71}$	0	0	0	0

Numerical Results For Univariate Case

Table 4: Comparison of Various Iterative Methods for $f_3(x)$

$f_3(x)$ $x_0 = -0.47$	NM	KM	CM	SM	SMNF1
n	9	5	5	5	5
$\lvert x_1 - \omega \rvert$	$5.20\ 10^{-4}$	$1.78\ 10^{-6}$	$8.70\ 10^{-7}$	$2.47\ 10^{-8}$	$2.39\ 10^{-7}$
$\lvert f_3(x_1) \rvert$	$8.54\ 10^{-4}$	$2.93\ 10^{-6}$	1.4310^{-6}	$4.06\ 10^{-8}$	$3.93\ 10^{-7}$
$\lvert x_2 - \omega \rvert$	$1.88\ 10^{-7}$	$3.03\ 10^{-23}$	$8.54\ 10^{-25}$	$1.39\ 10^{-32}$	$1.35\ 10^{-27}$
$\lvert f_3(x_2) \rvert$	$3.08\ 10^{-7}$	$4.99\ 10^{-23}$	$1.40\ 10^{-24}$	$2.29\ 10^{-32}$	$2.22\ 10^{-27}$
$\lvert x_3 - \omega \rvert$	$2.46\ 10^{-14}$	$2.54\ 10^{-90}$	$7.92\ 10^{-97}$	$1.41\ 10^{-129}$	$1.39\ 10^{-108}$
$\lvert f_3(x_3) \rvert$	$4.04\ 10^{-14}$	$4.17\ 10^{-90}$	$1.30\ 10^{-96}$	$2.31\ 10^{-129}$	$2.28\ 10^{-108}$
$\lvert x_4 - \omega \rvert$	$4.21\ 10^{-28}$	$1.24\ 10^{-358}$	$5.85\ 10^{-385}$	$1.47\ 10^{-517}$	$1.55\ 10^{-432}$
$\lvert f_3(x_4) \rvert$	$6.91\ 10^{-28}$	$2.04\ 10^{-358}$	$9.61\ 10^{-385}$	$2.42\ 10^{-517}$	$2.54\ 10^{-432}$
$\lvert x_5 - \omega \rvert$	$1.23\ 10^{-55}$	0	0	0	0
$\lvert f_3(x_5) \rvert$	$2.02\ 10^{-55}$	0	0	0	0
$\lvert x_6 - \omega \rvert$	$1.05\ 10^{-110}$	0	0	0	0
$\lvert f_3(x_6) \rvert$	$1.73\ 10^{-110}$	0	0	0	0

Numerical Results For Univariate Case

Table 5: Comparison of Various Iterative Methods for $f_4(x)$

$f_4(x)$ X_0=0.8	NM	KM	CM	SM	SMNF1
n	10	7	5	6	6
$\|x_1-\omega\|$	$1.45\ 10^{-2}$	$1.04\ 10^{-2}$	$1.79\ 10^{-4}$	$1.41\ 10^{-3}$	$1.41\ 10^{-3}$
$\|f_4(x_1)\|$	$1.54\ 10^{-2}$	$1.10\ 10^{-2}$	$1.89\ 10^{-4}$	$1.49\ 10^{-3}$	$1.49\ 10^{-3}$
$\|x_2-\omega\|$	$5.67\ 10^{-5}$	$1.58\ 10^{-9}$	$4.86\ 10^{-17}$	$3.92\ 10^{-12}$	$3.92\ 10^{-12}$
$\|f_4(x_2)\|$	$6.01\ 10^{-5}$	$1.67\ 10^{-9}$	$5.14\ 10^{-17}$	$4.15\ 10^{-12}$	$4.15\ 10^{-12}$
$\|x_3-\omega\|$	$8.54\ 10^{-10}$	$8.28\ 10^{-37}$	$2.63\ 10^{-67}$	$2.37\ 10^{-46}$	$2.37\ 10^{-46}$
$\|f_4(x_3)\|$	$9.04\ 10^{-10}$	$8.77\ 10^{-37}$	$2.78\ 10^{-67}$	$2.51\ 10^{-46}$	$2.51\ 10^{-46}$
$\|x_4-\omega\|$	$1.94\ 10^{-19}$	$6.16\ 10^{-146}$	$2.25\ 10^{-268}$	$3.17\ 10^{-183}$	$3.17\ 10^{-183}$
$\|f_4(x_4)\|$	$2.06\ 10^{-19}$	$6.52\ 10^{-146}$	$2.39\ 10^{-268}$	$3.35\ 10^{-183}$	$3.35\ 10^{-183}$
$\|x_5-\omega\|$	$1.00\ 10^{-38}$	$0.41\ 10^{-500}$	0	$1.01\ 10^{-730}$	$1.01\ 10^{-730}$
$\|f_4(x_5)\|$	$1.06\ 10^{-38}$	$2.00\ 10^{-582}$	0	$1.07\ 10^{-730}$	$1.07\ 10^{-730}$
$\|x_6-\omega\|$	$2.67\ 10^{-77}$	$4.14\ 10^{-500}$	0	0	0
$\|f_4(x_6)\|$	$2.83\ 10^{-77}$	0	0	0	0

Numerical Results For Univariate Case

Table 6: Comparison of Various Iterative Methods for $f_5(x)$

$f_5(x)$ $x_0 = 0.22$	**NM**	**KM**	**CM**	**SM**	**SMNF1**
n	10	D	D	8	6
$\lvert x_1 - \omega \rvert$	$2.09\ 10^{-1}$	$8.46\ 10^{0}$	$1.57\ 10^{0}$	$6.11\ 10^{-1}$	$2.49\ 10^{-1}$
$\lvert f_5(x_1) \rvert$	$2.44\ 10^{-1}$	$8.36\ 10^{-1}$	$1.51\ 10^{0}$	$7.77\ 10^{-1}$	$2.95\ 10^{-1}$
$\lvert x_2 - \omega \rvert$	$1.96\ 10^{-2}$	$11.00\ 10^{0}$	$1.24\ 10^{11}$	$1.41\ 10^{-1}$	$4.70\ 10^{-3}$
$\lvert f_5(x_2) \rvert$	$1.99\ 10^{-2}$	$9.97\ 10^{-1}$	D	$1.57\ 10^{-1}$	$4.70\ 10^{-3}$
$\lvert x_3 - \omega \rvert$	$3.53\ 10^{-4}$	$1.12\ 10^{9}$	D	$7.90\ 10^{-4}$	$2.14\ 10^{-9}$
$\lvert f_5(x_3) \rvert$	$3.53\ 10^{-4}$	$9.96\ 10^{-1}$	D	$7.91\ 10^{-4}$	$2.14\ 10^{-11}$
$\lvert x_4 - \omega \rvert$	$1.25\ 10^{-7}$	$1.12\ 10^{9}$	D	$2.28\ 10^{-12}$	$9.6\ 10^{-35}$
$\lvert f_5(x_4) \rvert$	$1.25\ 10^{-7}$	$1.70\ 10^{-1}$	D	$2.28\ 10^{-12}$	$9.6\ 10^{-35}$
$\lvert x_5 - \omega \rvert$	$1.56\ 10^{-14}$	$1.12\ 10^{9}$	D	$1.59\ 10^{-46}$	$3.89\ 0^{-136}$
$\lvert f_5(x_5) \rvert$	$1.56\ 10^{-14}$	$9.4\ 10^{-6}$	D	$1.59\ 10^{-46}$	$3.89\ 10^{-136}$
$\lvert x_6 - \omega \rvert$	$2.44\ 10^{-28}$	$1.12\ 10^{9}$	D	$3.82\ 10^{-183}$	$9.72\ 10^{-266}$
$\lvert f_5(x_6) \rvert$	$2.44\ 10^{-28}$	$3.91\ 10^{-27}$	D	$3.82\ 10^{-183}$	$9.72\ 10^{-266}$

*D stands for divergen

Numerical Results For Multivariate Case

Table 7: Test Functions With Their Exact Roots and Initial Guesses

	Numerical Example	Exact Zero	Starting vector
Example 1	$f = x - y - 19$ $g = \frac{y^3}{6} - x^2 - 17$	$\mathbf{W} = \begin{bmatrix} 5.000000 \\ 6.000000 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 5.1 \\ 6.1 \end{bmatrix}$
Example 2	$f = \cos y - \sin x$ $g = z^x - \frac{1}{y}$ $h = e^x - z^2$	$\mathbf{W} = \begin{bmatrix} 0.909569 \\ 0.661227 \\ 1.575834 \\ 1.575834 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 1.0 \\ 0.5 \\ 1.5 \end{bmatrix}$
Example 3	$f = xy + z(x+y)$ $g = wy + z(w+y)$ $h = wx + z(w+x)$ $i = wx + wy + xy - 1$	$\mathbf{W} = \begin{bmatrix} 0.577350 \\ 0.5577350 \\ 0.577350 \\ -0.288675 \end{bmatrix}$	$\mathbf{X}^{(0)} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.2 \end{bmatrix}$

Numerical Results For Multivariate Case

Table 8: Comparison of Various Iterative Methods for Solving System of Nonlinear Equations

Numerical Example		**NM**	**KM**	**BM**	**SMNF2**
Example 1	Iterations	7	5	4	4
	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$4.98\ 10^{-115}$	$1.18\ 10^{-376}$	$2.59\ 10^{-103}$	$7.10\ 10^{-108}$
	$\left\|\mathbf{F}(\mathbf{X}^{(k-1)})\right\|_\infty$	$3.31\ 10^{-115}$	$2.13\ 10^{-375}$	$4.64\ 10^{-102}$	$1.27\ 10^{-106}$
Example 2	Iterations	9	6	6	6
	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$6.42\ 10^{-108}$	$2.76\ 10^{-210}$	$1.00\ 10^{-299}$	$1.18\ 10^{-356}$
	$\left\|\mathbf{F}(\mathbf{X}^{(k-1)})\right\|_\infty$	$6.46\ 10^{-108}$	$2.79\ 10^{-210}$	$1.02\ 10^{-299}$	$1.23\ 10^{-356}$
Example 3	Iterations	8	5	5	5
	$\left\|\mathbf{X}^{(k)}-\mathbf{X}^{(k-1)}\right\|_\infty$	$4.40\ 10^{-145}$	$1.30\ 10^{-198}$	$3.49\ 10^{-238}$	$1.02\ 10^{-256}$
	$\left\|\mathbf{F}(\mathbf{X}^{(k-1)})\right\|_\infty$	$4.51\ 10^{-145}$	$1.30\ 10^{-198}$	$4.02\ 10^{-238}$	$1.02\ 10^{-256}$

Numerical Results For Multivariate Case

Table 9: Comparison of Computational Cost of Various Iterative Methods for Solving Systems

Methods	Convergence Order	Number of Functional Evaluations	Total Cost of Method
BM	4	$2n^2+n$	$\frac{13}{3}n^3+12n^2+4n$
KM	4	$2n^2+n$	$\frac{17}{3}n^3+7n^2+6n$
CM	4	$2n^2+n$	$\frac{11}{3}n^3+10n^2+4n$
SMNF2	4	$2n^2+n$	$\frac{13}{3}n^3+12n^2+4n$

Conclusion

CONCLUSION

- The Method has Optimal Fourth Order Convergent
- Comparable With All Univariate Fourth Order Methods of This Domain
- Comparable With All Multivariate Fourth Order Methods of This Domain in Terms of Computaional Cost given in Table 9

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SPECIAL THANKS TO

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