Dynamical Behaviour of Root Finding Iterative Methods

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INTRODUCTION

- In 17th century, Newton proposed a method for calculating approximate roots of polynomials.
- A method of convergence order 2^n is called Optimal if it requires *n-1* function evaluations.
- It is known that any complex polynomial p of degree n having n roots, according to fundamental theorem of algebra, can be uniquely defined by its coefficients:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

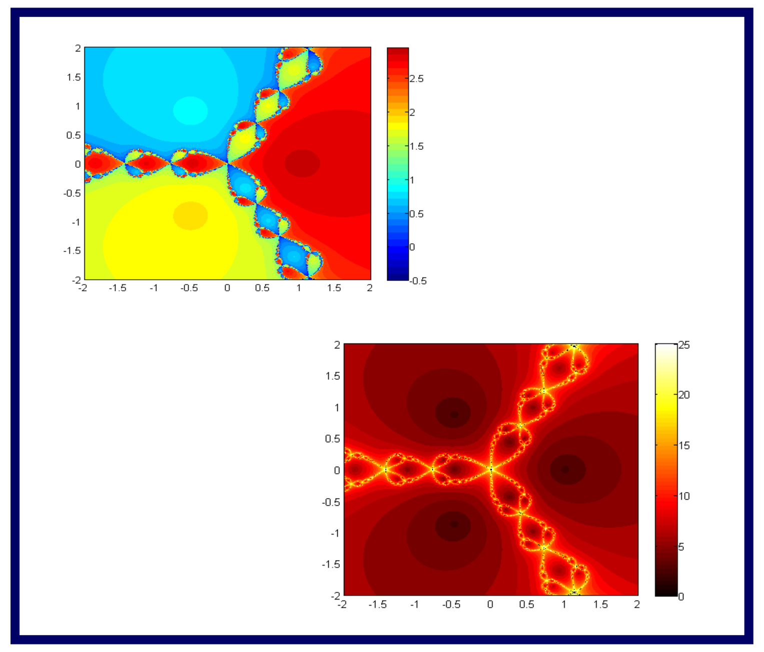
- Basins of Attraction define visualization process of approximating the root of complex polynomials using fractal and non-fractal images.
- Root finding methods can be applied to polynomials p as a result polynomiographs are generated.
- An individual image is called a Polynomiograph.

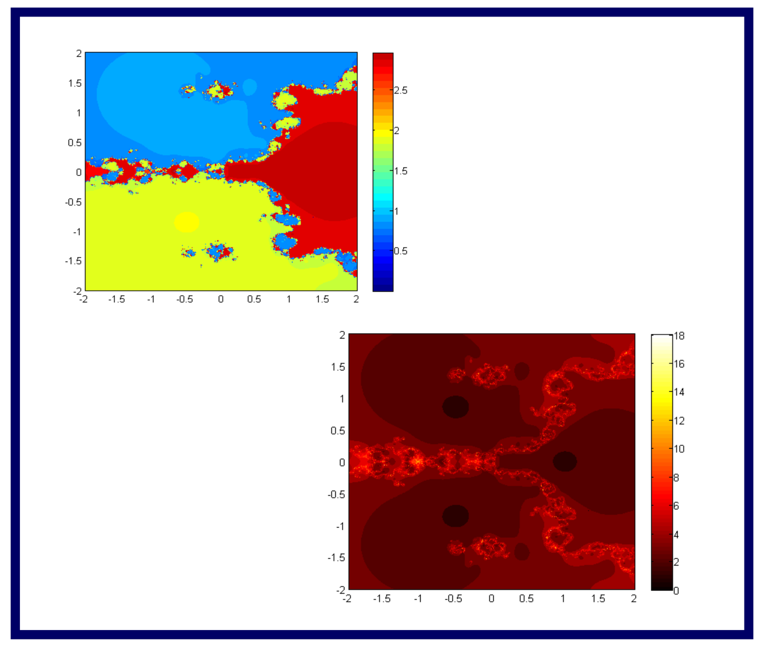
METHODOLOGY

- We find here the roots of complex polynomials p_n, $n\geq 1$.
- We use MATLAB to generate basins of attraction.
- We take a square box of [-2, 2]×[-2, 2].
- The colors of polynomiographs are based on number of iterations as well as number of roots.
- We use two techniques:
  In the first technique :
  i. For every initial guess x_0 , a specific color is assigned according to its convergence to an exact root.
  ii. Dark blue is assigned for the divergence of the method.
  iii. Color map is chosen as "JET".
  In the second technique:
  i. Each initial guess is assigned by a color depending upon to the number of iterations for the method to converge to any of the root of the given function.
  ii. Color map is chosen as "HOT".
  iii. Divergence for any initial guess is displayed by "Black" color.

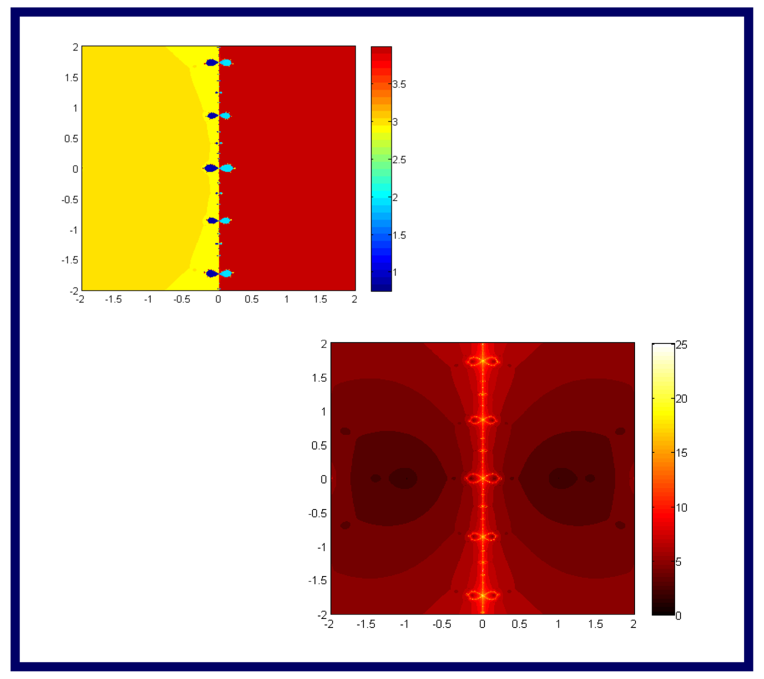
BASINS OF ATTRACTION

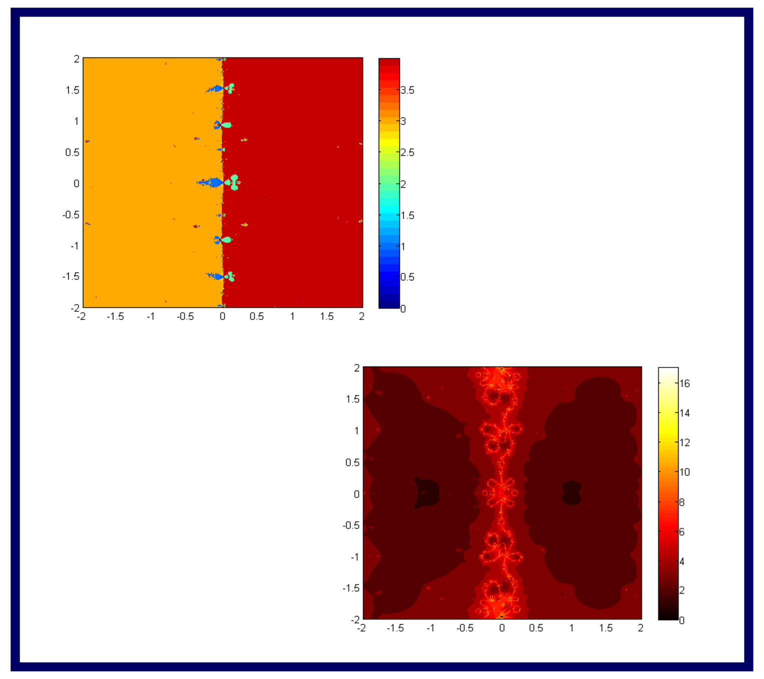
Newton's Method $P_3(z)=z^3-1$ Our Method



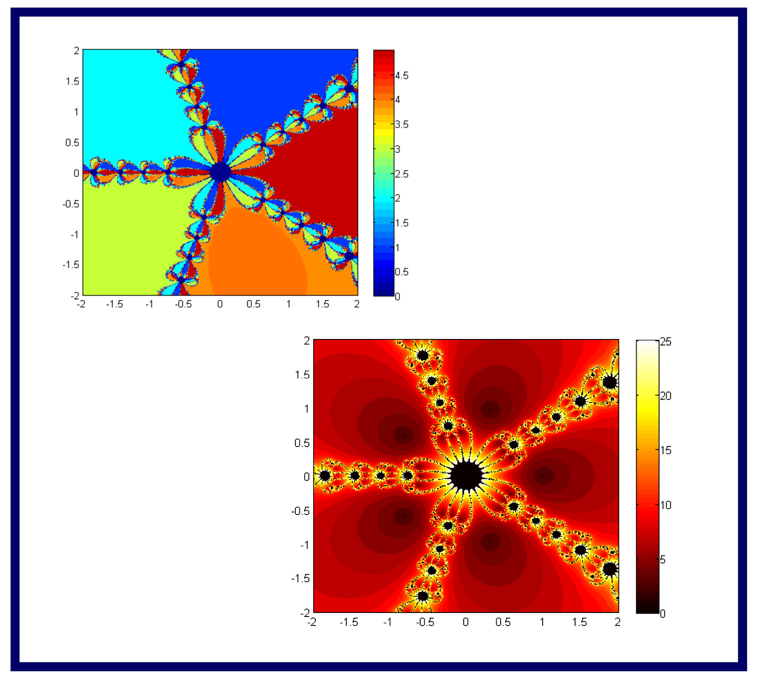


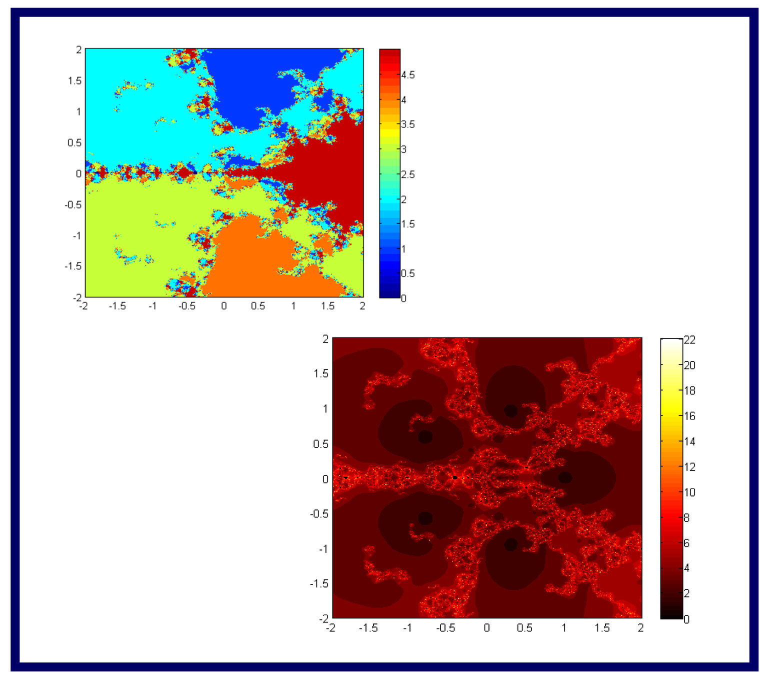
Newton's Method $P_4(z)=z^4-10z^2+9$ Our Method





Newton's Method b $P_5(z)=z^5-1$ Our Method





NUMERICAL METHODS

Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

is a with-derivative quadratically convergent root-finding iterative method with 2 function evaluation.

Our Optimal Sixteenth Order Method:

$$x_{n+1} = x_n - \frac{f(x_n)(m_1+m_2+m_3+m_4)}{m_1 f[w_1,x_n] + m_2 f[w_2,x_n] + m_3 f[w_3,x_n] + m_4 f[w_4,x_n]}$$

$$m1 = f(w_2)f(w_3)f(w_4)\{-(w_3-x_n)(w_4-x_n)(w_4-w_3) + (w_2-x_n)(w_4-x_n)(w_4-w_2)\} - (w_2-x)(w_3-x_n)h_1(f(w_4))$$

$$m2 = f(w_1)f(w_3)f(w_4)\{(w_3-x)(w_4-x)(w_4-w_3) - (w_1-x_n)(w_4-x)(w_4-w_1)\} - (w_1-x)(w_3-x_n)h_2(f(w_4))$$

$$m3 = f(w_1)f(w_2)f(w_4)\{-(w_2-x_n)(w_4-x)(w_4-w_2) + (w_1-x_n)(w_4-x_n)(w_4-w_1)\} - (w_1-x_n)(w_2-x_n)h_3(f(w_4))$$

$$m4 = f(w_1)f(w_2)f(w_3)\{(w_2-x_n)(w_3-x_n)(w_3-w_2) - (w_1-x_n)(w_3-x_n)(w_3-w_1)\} + (w_1-x_n)(w_2-x_n)h_3(f(w_3))$$

is a without derivative sixteenth order convergent optimal iterative method with 4 function evaluations.

CONCLUSIONS

We compare the results of our newly constructed method with Newton's method from dynamical point of view. Two types of attraction basins are given in all figures. Color maps with each figure show the root to which an initial guess converges and the number of iterations in which the convergence occur. One can easily see that the appearance of darker region shows that our method consumes less number of iterations as compared to Newton's method. It even converges where Newton's method diverges.

REFERENCES

1. B. Kalantari, Polynomial root finding and Polynomiography, World Scientific Publishing, 5 Toh Tuck Link, Singapore, 2009.
2. Fiza Zafar, Nusrat Yasmin, Saima Akram and Moin-ud-din Junjua, A General Class of Derivative Free Optimal Root Finding Methods Based on Rational Interpolation, The Scientific World Journal, Volume 2014 (2014), Article ID 934260, 16 pages.