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An elementary proof of Euler's formula using Cauchy's method

Jean-Paul Brasselet [a,*], Nguyễn Thị Bích Thủy [b,*]

[a] *CNRS I2M and Aix-Marseille Université, Marseille, France*
[b] *UNESP, Universidade Estadual Paulista, "Júlio de Mesquita Filho", São José do Rio Preto, Brazil*

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A B S T R A C T

The use of Cauchy's method to prove Euler's well-known formula is an object of many controversies. The purpose of this paper is to prove that Cauchy's method applies for convex polyhedra and not only for them, but also for surfaces such as the torus, the projective plane, the Klein bottle and the pinched torus.

0. Introduction

Euler's formula says that for any convex polyhedron the alternating sum

$$n_0 - n_1 + n_2, \tag{1}$$

is equal to 2, where the numbers n_i are respectively the number of vertices n_0, the number of edges n_1 and the number of triangles n_2 of the polyhedron. There are many controversies about the paternity of the formula, also about who gave the first correct proof.

In section 1, we provide information about the history of the formula as well as about the first topological proof given by Cauchy. Some authors criticize Cauchy's proof, saying that the proof needs deep topological results that were proved after Cauchy's time: "*Não se pode, portanto, esperar obter uma demonstração elementar do Teorema de Euler, com a hipótese de que o poliedro é homeomorfo a uma esfera, como fazem*

* Corresponding authors.
E-mail addresses: jean-paul.brasselet@univ-amu.fr (J.-P. Brasselet), bich.thuy@unesp.br (N.T.B. Thủy).

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Hilbert-Cohn Vossen e Courant-Robbins" (Lima, [26]).[1] Notice that the proof provided by Hilbert-Cohn Vossen [18] and Courant-Robbins [6] is the one of Cauchy.

In section 2, we provide an elementary proof which shows that only with a lifting technique and the use of sub-triangulations, Cauchy's proof works without using any other result. More precisely, considering a triangulated polygon in the plane, with possible identifications of the simplices on its boundary, we prove that the alternating sum (1) of the polygon is equal to the one of its boundary plus 1 (Theorem 2.1). The idea of our proof is, starting from the hole formed by the removal of a simplex, to extend the hole by successive puddles. The process is illustrated by the construction of a suitable pyramid. A direct consequence of the theorem is an elementary proof of Euler's formula using only Cauchy's method.

As applications of our Theorem 2.1, in section 3, we also use these tools to prove that for a triangulable surface $\mathcal{S}$ like the torus, the projective plane, the Klein bottle and even for the pinched torus, the alternating sum (1) does not depend on the triangulation of the surface. To be completely honest, for applications (other than the sphere) in section 3, we also use the idea of "cutting" surfaces that, in general, was introduced by Oswald Veblen in a seminar in 1915 (see [1]). Of course, one can ask why we do not apply Theorem 2.1 and the same reasoning to all (smooth) orientable and non-orientable surfaces. The reason is very simple: we want to provide proofs that it was possible to do at the time of Cauchy. It is only in 1925 that T. Radó [31] proved the triangulation theorem for surfaces, that was more or less assumed in Cauchy's time. The classification theorem for compact surfaces and the representation by the "normal" form was proved for the first time in a rigorous way by H.R. Brahana [1] (1921). It is clear that using our Theorem 2.1 and the representation of surfaces under the normal form, we immediately obtain the Euler-Poincaré characteristic of any compact surface. However, that is like a dog biting his own tail. That it is the reason we do not present the result for surfaces in general but only what is possible to do with Cauchy's method for some elementary surfaces.

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1. History

1.1. Before Cauchy

The name "Euler's formula" comes from an announcement of Leonhard Euler in November 14th of 1750 in a letter to his friend Goldbach [12] of the following result:

Theorem 1.1. *Let K be a convex polyhedron, with n_0 vertices, n_1 edges and n_2 two-dimensional polygons, then*

$$n_0 - n_1 + n_2 = 2. \tag{2}$$

There are many different possible definitions of polyhedra. The discussion concerns the dimension of a polyhedron: Is a polyhedron a solid object of dimension three or only its surface? In this paper, we call "polyhedron" the three dimensional solid figure. A polyhedron is a figure constructed by polygons in such a way that each segment is the common face of exactly two 2-dimensional polygons and each vertex is the common face of at least three segments (see [32], Chapter 2 for discussion).

There are many questions and controversies about Euler's formula. Here, let us discuss the two following questions: Was Euler the first mathematician stating the formula? Who provided the first proof of the formula?

Let us discuss the first question: *Who was the first to state the formula?*

[1] *Therefore, we cannot expect to obtain an elementary proof of Euler's theorem, with the hypothesis that the polyhedron is homeomorphic to a sphere, as Hilbert-Cohn Vossen and Courant-Robbins did.*

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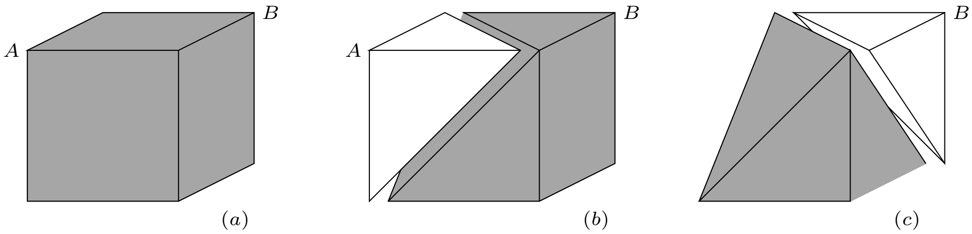

Fig. 1. Euler's proof: Successive elimination of a vertex as well as the pyramid of which it is the vertex.

Some authors (see [15], §3.12; [25], p. 90) write that it is possible that Archimedes (∼ 287 AC, ∼ 212 AC) already knew the formula. Some authors say that the formula was known to Descartes (1596 – 1650). In fact, Descartes, in a manuscript [10], *"De solidorum elementis"*, proved the following result:

Theorem 1.2. *The sum of the angles of all the 2-dimensional polygons of a convex polyhedron is equal to* $2(n_0 - 2)\pi$.

Proof that "formula (2) is equivalent to Theorem 1.2". Let us denote by $i = 1, \dots, n_2$ the 2-dimensional faces of a convex polyhedron. For each face i, let us denote by k_i the number of vertices, which is also the number of edges of the face. We use, for each face, the following property: In a convex polygon with k_i edges, the sum of all the angles equals $(k_i - 2)\pi$. Since each edge of the polyhedron appears in two faces of the polyhedron, then $\sum_{i=1}^{n_2} k_i = 2n_1$. Hence the sum of the angles of all the faces of the polyhedron equals $\sum_{i=1}^{n_2}(k_i - 2)\pi$, that is $(2n_1 - 2n_2)\pi$. We obtain equivalence between Theorem 1.2 and formula (2). □

Descartes did not publish his manuscript. The original version of the manuscript suffered some accidents, in particular an immersion in the Seine river in Paris (see [16], [7]). A copy was rediscovered in 1860 among papers left by Leibnitz (1646-1716) and published by E. de Jonquières (see [8], [9]).

Some authors say that Descartes discovered the formula (2) as an application of his Theorem 1.2. This is a reason why sometimes the formula (2) is called the "*Descartes-Euler formula*". Other authors, for example, Malkevich [27], affirmed that "*Though Descartes did discover facts about 3-dimensional polyhedra that would have enabled him to deduce Euler's formula, he did not take this extra step. With hindsight it is often difficult to see how a talented mathematician of an earlier era did not make a step forward that with today's insights seems natural, however, it often happens.*" However, we know that some of Descartes' papers disappeared, so nobody can decide if Descartes knew the formula or not and the response to the first question is not known.

Let us now discuss the second question: *Who was the first to provide a correct proof of formula (2)?*

Two years after writing the formula, Euler provided a proof [13,14] which consisted of removing step by step each vertex of the polyhedron together with the pyramid of which it is the vertex.

Here we provide the example of the cube, copied from [32]: In Fig. 1 (b), one eliminates the vertex A as well as the (white) pyramid of which A is a vertex. This operation does not change the alternating sum $n_0 - n_1 + n_2$. In Fig. 1 (c) we perform the same process in order to eliminate the vertex B, and so on, till we obtain a tetrahedron. For the tetrahedron, one has $n_0 - n_1 + n_2 = 2$, so we obtain the formula.

But this proof is not correct. In the book [32], Richeson provides a clear description of Euler's proof, as well as the problems with the proof. According to Richeson, these problems were solved by Samelson and by Francese and Richardson ([33,17]).

The first correct proof was provided by Legendre [23] in the first edition of his book *Éléments de Géométrie* (1794) (see [22] and also [32], Chapter 10 for a presentation of Legendre's proof). Legendre's argument was geometric in the same way as the proof of Descartes for Theorem 1.2. The only difference between Descartes' argument and Legendre's argument is that Descartes used the sphere presentation of the polyhedron (polar polyhedron), while Legendre used the polyhedron itself. The passage of the polyhedron K (Legendre) to the polar polyhedron of K (Descartes) makes a permutation of n_0 and n_2, without modifying n_1. This is the reason why some authors say that the proof of Euler's formula (2) should be called "Descartes-Legendre's proof".

1.2. Cauchy's time: Cauchy's method - The first combinatorial and topological proof

In February of 1811, then 22 years old, Cauchy, who was already an engineer of *Ponts et Chaussées*, gave a talk entitled *Recherches sur les polyèdres* at the *École Polytechnique*, in Paris. This talk was published in 1813 in the *Journal de l'École Polytechnique* ([3], [4]), as the first combinatorial and topological proof of Euler's formula (2). This nice proof of Cauchy is included in many books (see, in particular, [32], Chapter 12 for a presentation with comments).

The first step of Cauchy's proof is to construct a planar representation.

Definition 1.3. A planar representation of a compact and without boundary surface $\mathcal{S}$ is a triple (K, K_0, φ) where:

(1) K is a 2-dimensional polygon in $\mathbb{R}^2$,
(2) the segments and vertices of the boundary of K are named and oriented with possible identifications. We denote by K_0 the boundary of K with the given identifications,
(3) $\varphi : |K| \to \mathcal{S}$ is a homeomorphism of the geometrical realization of K (taking care of identifications of the segments and vertices on the boundary K_0) onto $\mathcal{S}$.

It seems that Cauchy was the first person to use the idea of planar representation. We now present Cauchy's proof.

Proof. Given a convex polyhedron $\widehat{K}$, we choose a 2-dimensional face $\mathcal{P}$ of the polyhedron. We remove this face. The first idea of Cauchy is to construct the associated planar representation K of the polyhedron with respect to the choice of the removed face. Lakatos [21] explains Cauchy's construction as the following way: Put a camera above the removed face of the polyhedron, the planar representation will appear as the photograph. Notice that this idea of planar representation is similar to stereographic projection.

The hole formed by the removed face appears outside in the planar representation (see the blue part in Fig. 3).

The next step of Cauchy's proof is to define a triangulation of K by a subdivision of all the polygons. Notice that, in the triangulating process, the alternating sum $n_0 - n_1 + n_2$ does not change. The boundary of the hole consists of edges with the following property: Each edge is a face of a triangle that has only this edge as the common edge with the hole. For example, in Fig. 4 (a), the triangle σ has the edge τ as the common edge with the hole. The extension of the hole consists of two operations that we describe in what follows.

The first "operation I" consists of removing from the polyhedron K such a triangle σ together with its corresponding edge τ and then the hole is extended. Operation I do not change the sum $n_0 - n_1 + n_2$, because n_0 does not change while n_1 and n_2 decrease by 1.

When a situation like the one of Fig. 4 (b) appears, where one triangle (here σ) has two common edges τ_1 and τ_2 with the hole, we use "operation II" which consists of removing from the polyhedron K the triangle σ together with the two edges τ_1 and τ_2 and the vertex a that is the common vertex of τ_1 and τ_2. Then, the hole is extended.

Operation II also does not change the sum $n_0 - n_1 + n_2$, since n_0 and n_2 decrease by 1 and n_1 decreases by 2.

If we take care of keeping the boundary homeomorphic to a circle, then the hole is extended, using the two operations above, until we have only one triangle. In this triangle, we have $n_0 - n_1 + n_2 = 3 - 3 + 1 = 1$.

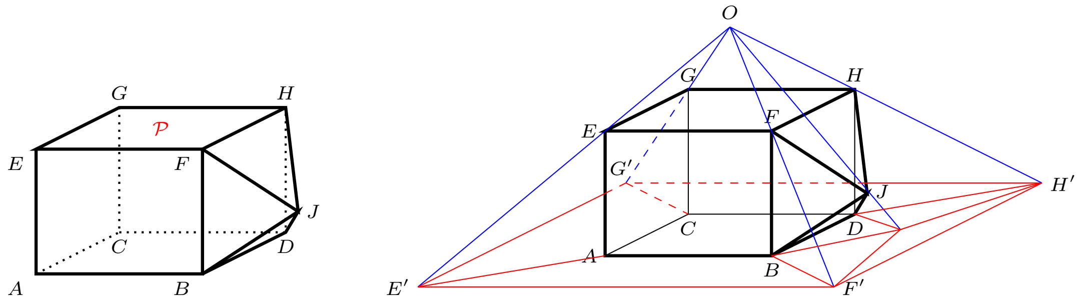

Fig. 2. Planar representation according to Cauchy. The pyramid with vertex O is supposed translucent.

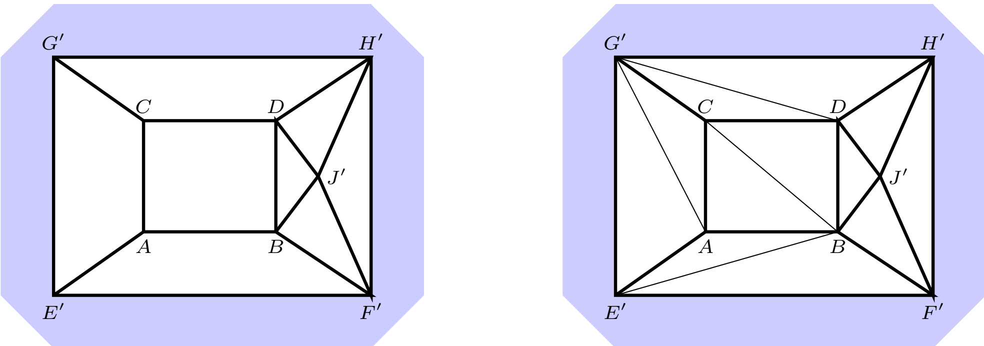

Fig. 3. The polygon K and its triangulation. The hole is in blue. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

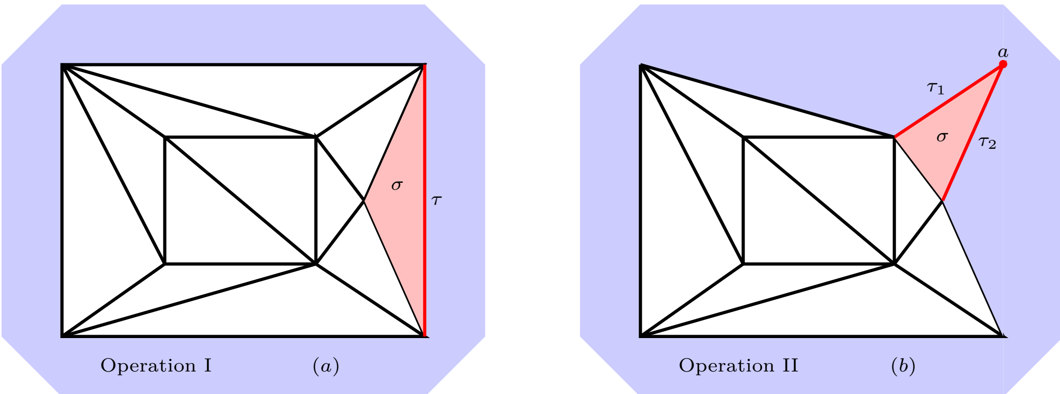

Fig. 4. The two Cauchy operations.

Since we removed an (open) polygon at the beginning, we have already $+1$ in the sum $n_0 - n_1 + n_2$. Hence for any convex polyhedron, we have $n_0 - n_1 + n_2 = +2$. $\square$

1.3. After Cauchy

Some authors, in particular Lakatos [21], criticize Cauchy's proof. In his book (see [21], pages 11 and 12), Lakatos provided a counter-example to Cauchy's process. Here, we adapt Lakatos' counter-example to our example in Fig. 2. Extending the hole by removing triangles in the indicated order in Fig. 5 (a), we use operations I and II of Cauchy until the ninth triangle. If we remove the tenth triangle, the hole disconnects the rest of the figure (see Fig. 5 (b)): The eleventh and twelfth triangles are no longer connected. Moreover, the boundary of the hole is no longer homeomorphic to a circle. Finally, we observe that if we remove the

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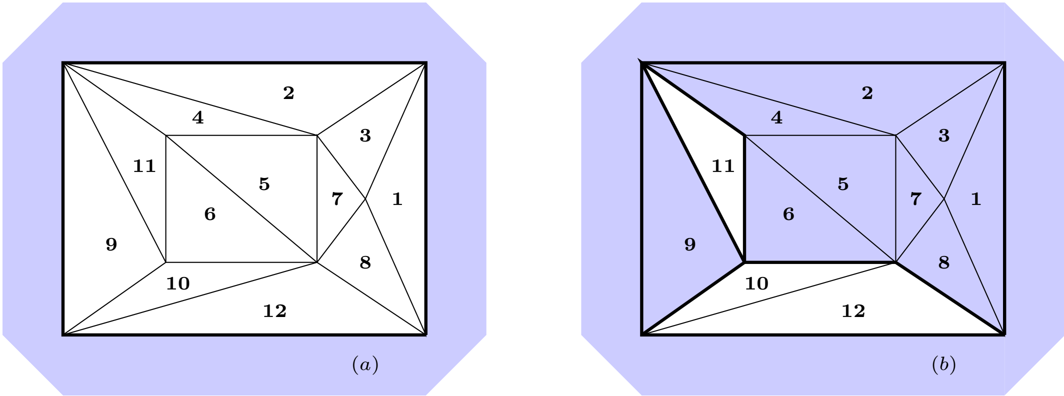

Fig. 5. Removal order according to Lakatos.

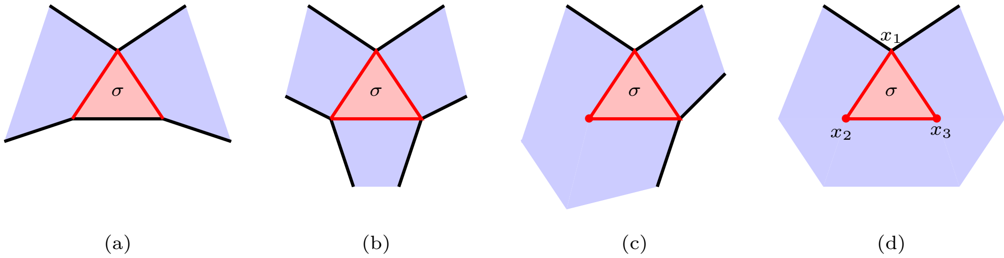

Fig. 6. The counter-examples of Lima: The hole B is in blue and the simplices to be removed are in red and pink.

tenth triangle, we do not remove any vertex, but we remove two edges and one triangle, therefore the sum $n_0 - n_1 + n_2$ is not preserved.

Hence, we need to be very careful concerning the order of the removal of the triangles since it can happen that the hole disconnects the polyhedron K. Moreover, the boundary of the hole is no longer a curve homeomorphic to a circle because it has multiple points.

In the paper [26], Lima formalized the arguments of Lakatos and described the situation of Fig. 6. In figures (a), (b) and (c), the extension of the hole, obtained by the removal of the triangle σ from the polyhedron K, on the one hand, changes the sum $n_0 - n_1 + n_2$ and, on the other hand, disconnects the polyhedron K. Lakatos' example corresponds to the situation (a) in figure of Lima. In the examples of Lima, the boundary of the hole is a curve with multiple points.

We observe that figure (d) is also a case where the boundary is a curve having multiple points. However, the sum $n_0 - n_1 + n_2$ is preserved when we remove the triangle σ from K since we remove two vertices x_2 and x_3, three edges (x_1, x_2), (x_2, x_3), (x_1, x_3) and the triangle σ. It seems that Lima did not realize that this case is admissible. The situation of figure (d) can be also used in the process of Cauchy. In this paper, we call this operation "operation III". This operation appeared also in [5].

Note that with operation III, Lakatos' example is no longer a counter-example: come back to Fig. 5 (a), after removing the ninth triangle, one can remove the twelfth triangle by operation II, then remove the tenth triangle by operation III. We have only the eleventh triangle for which $n_0 - n_1 + n_2 = +1$ so we are done!

Some authors suggest a strategy to define an order of removal of the triangles that allows the use of Cauchy's method to obtain the result. For example, Kirk [20] suggests the following strategy: "*There are two important rules to follow when doing this. Firstly, we must always perform* [Operation II] *when it is possible to do so; if there is a choice between* [Operation I] *and* [Operation II] *we must always choose* [Operation II]*. If we do not, the network may break up into separate pieces. Secondly, we must only remove faces one at a time.*" We provide, in Fig. 7 (b), an example whose the process follows the rules of the

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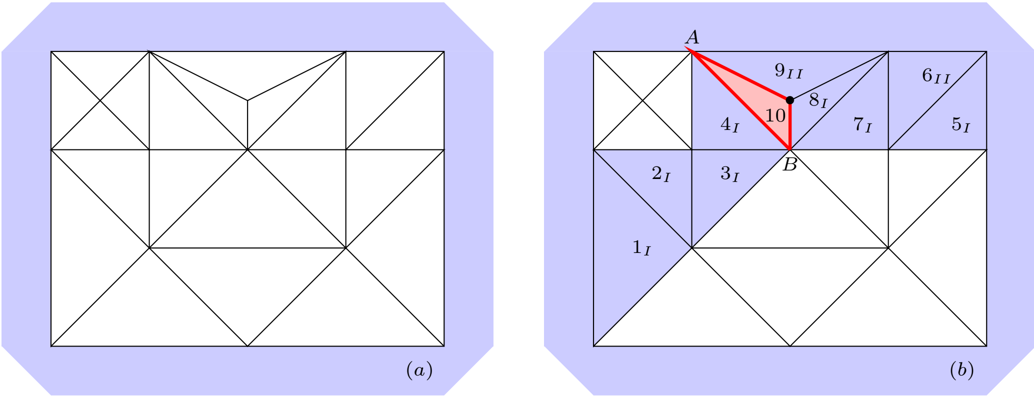

Fig. 7. A counter-example to the “strategy” suggested by Kirk. Triangles are removed in the order of numbers. The index I or II below each number means that the triangle is removed using the corresponding operation I or II, respectively. After removing the seventh triangle, the boundary of the hole is no longer homeomorphic to a circle.

strategy defined by Kirk but the process disconnects the polyhedron. So these rules are not sufficient. It is easy to build an example showing that they are not necessary.

The example provided in Fig. 7 is also a counter-example showing that even with operation III, Cauchy’s process does not work with the given order of removal. In fact, after removing the seventh triangle, we can continue untill the tenth triangle, but we cannot remove it because in that case we remove one vertex, three segments and one triangle. Notice that the vertices A and B do not belong to the hole.

After Cauchy, many authors proposed alternating proofs of Euler’s formula (2), using original arguments. See the site of Eppstein [11] containing 20 different proofs, using tools that appeared only after Cauchy’s time. In particular, some proofs use the Jordan Lemma (Jordan [19], 1866). In fact, we will see that Jordan curves will appear in the proof of our Theorem 2.1 as an artifact.

However, to our knowledge, no one has given a strategy for removing the triangles that allows only the use of Cauchy’s method and tools known in Cauchy’s time. That is the goal of our paper.

1.4. Generalization of the formula (2)

The formula (2) was extended by many authors, in particular by Lhuilier [24], first for orientable surfaces of genus g, as follows:

$$n_0 - n_1 + n_2 = 2 - 2g. \tag{1.4}$$

In the non-orientable case, the formula is given by (see [28]):

$$n_0 - n_1 + n_2 = 2 - g.$$

The general result was provided by Poincaré [29,30] who proved that, for any triangulation of a polyhedron X of dimension k, where n_i is the number of simplices of dimension i, the sum

$$\chi(X) = \sum_{i=0}^{k} (-1)^i n_i \tag{3}$$

does not depend on the triangulation of X. This sum is called the Euler-Poincaré characteristic of X.

Due to the dimension convention for polyhedra in section 1.1, “Euler’s formula” would be better written as the Euler-Poincaré characteristic of the convex 3-dimensional polyhedral in the form

$$n_0 - n_1 + n_2 - n_3 = +1.$$

Of course, n_3 is $+1$ anyway, but this form of Euler’s formula seems more suitable.

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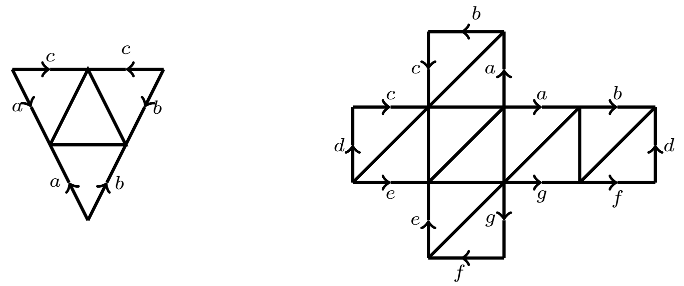
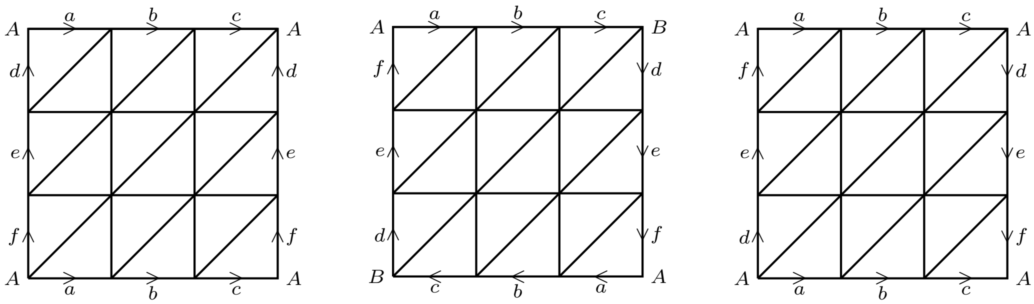


Fig. 8. Planar representations of the sphere.



The torus The projective plane The Klein bottle

Fig. 9. Planar representations of the torus, the projective plane and the Klein bottle.

2. Cauchy's method in the proof of Euler's formula

The classical planar representations of surfaces such as the sphere, the torus, the projective plane and the Klein bottle are examples of the following situation: The surface is homeomorphic to the geometric representation of a polygon K, itself homeomorphic to a disc D, such that there are possible identifications of simplices on the boundary K_0. (See Figs. 8 and 9.)

In this section, we prove the following theorem using only Cauchy's method (section 1.2) and sub-triangulations:

Theorem 2.1. *Let K be a triangulated polygon in $\mathbb{R}^2$, homeomorphic to a disc D, with possible identifications of simplices on the boundary K_0 of K. We have*

$$\chi(K) = \chi(K_0) + 1.$$

We emphasize that the important point of our proof is that, from the given triangulation, we provide a sub-triangulation such that we can prove the theorem using only Cauchy's method (section 1.2), without using other tools.

Proof of Theorem 2.1 using only Cauchy's method. Given a polygon K triangulated and homeomorphic to a disc in $\mathbb{R}^2$ with possible identifications of the simplices on the boundary K_0 of K, the proof consists of six steps, as follows.

1) Step 1: The first step is to construct a "lifting" of K into a pyramidal shape. Here, we call the "pyramid" only the surface (dimension 2) of the pyramid, *i.e.* the union of the faces of the pyramid without the base.

We can assume that the origin 0 of $\mathbb{R}^2$ lies inside a 2-dimensional simplex σ_0 in the interior of the polygon K (see Fig. 10).

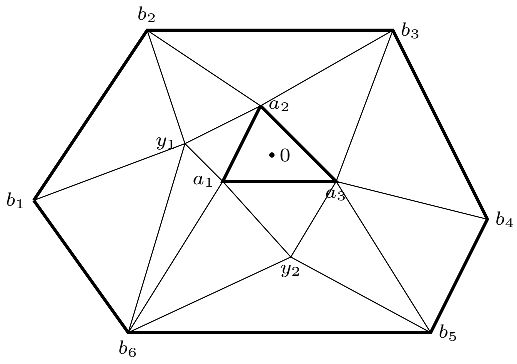


Fig. 10. The triangulation of the polygon K.

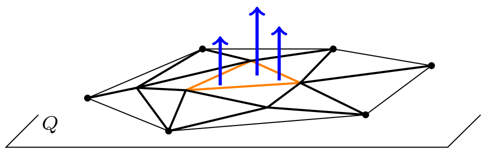


Fig. 11. The lifting on the polygon.

Let us consider the Euclidean metric in $\mathbb{R}^2$. We can assume that the distances from the origin to the vertices of the triangulation are different, otherwise a small perturbation will not change the structure of the simplicial complex and the proof of the theorem can be processed in the same way.

Let us denote by a_1, a_2, a_3 the vertices of σ_0 and by $b_1, \dots, b_k$ the vertices of the triangulation of K_0. The other vertices are denoted by the following way: We call y_1 the vertex nearest to the origin 0 and $y_2, \dots, y_n$ the vertices in the increasing order of distances from 0 (see Fig. 10).

We construct a pyramid Π in $\mathbb{R}^3$ lying above K by fixing the boundary K_0 as the base of the pyramid in the horizontal plane $Q = \mathbb{R}^2$ in $\mathbb{R}^3$. For $i = 0, \dots, n+1$, we consider the planes P_i parallel to Q having distance $n - i + 1$ relative to the base plane Q in $\mathbb{R}^3$. Now, for $i = 1, \dots, n$, we denote by x_i the orthogonal projection of the point y_i to the plane P_i. In the plane P_0, we denote by u_1, u_2, u_3 the orthogonal projections of the points a_1, a_2, a_3 (see Fig. 12 (a)).

The triangulation of the polygon induces a triangulation L on the pyramid Π lifting each simplex $[b_i, y_j]$ to $[b_i, x_j]$, each $[y_i, y_j]$ to $[x_i, x_j]$ and each $[y_i, a_j]$ to $[x_i, u_j]$. In the same way, we also lift the 2-dimensional simplices.

2) Step 2: Let us construct a sub-decomposition L' of the triangulation L, such that the intersections of the planes with the pyramid are triangulated in the following way: Let us define new vertices of L' as the intersection of 1-dimensional simplices of L with the planes P_i. In the same way, we define also new 1-dimensional simplices of L' as the intersections of 2-dimensional simplices of L with the planes P_i. The decomposition L' of the pyramid contains vertices, edges (1-dimensional simplices), and faces which can be triangles or quadrilaterals. The sum $n_0 - n_1 + n_2$ is the same for the triangulation L and the decomposition L' (see Fig. 12 (b)).

3) Step 3: Let us define a sub-triangulation L'' of L in the following way: each quadrilateral is divided into two triangles. The sum $n_0 - n_1 + n_2$ is the same for the triangulations L and L'' (see Fig. 13).

4) Step 4: We will prove that the intersection of L'' with each plane P_i is a curve homeomorphic to a circle. First, we show that the projection of L'' to the plane Q provides a sub-triangulation K'' of the polygon K (Fig. 14). In fact, by construction, since there is no vertical edges in the pyramid, then the orthogonal projection π of the pyramid to Q is a bijection between the triangulations L'' and K''. Notice that each vertex of K'' corresponds to a vertex of L'' and, in the same way, for the edges and the triangles

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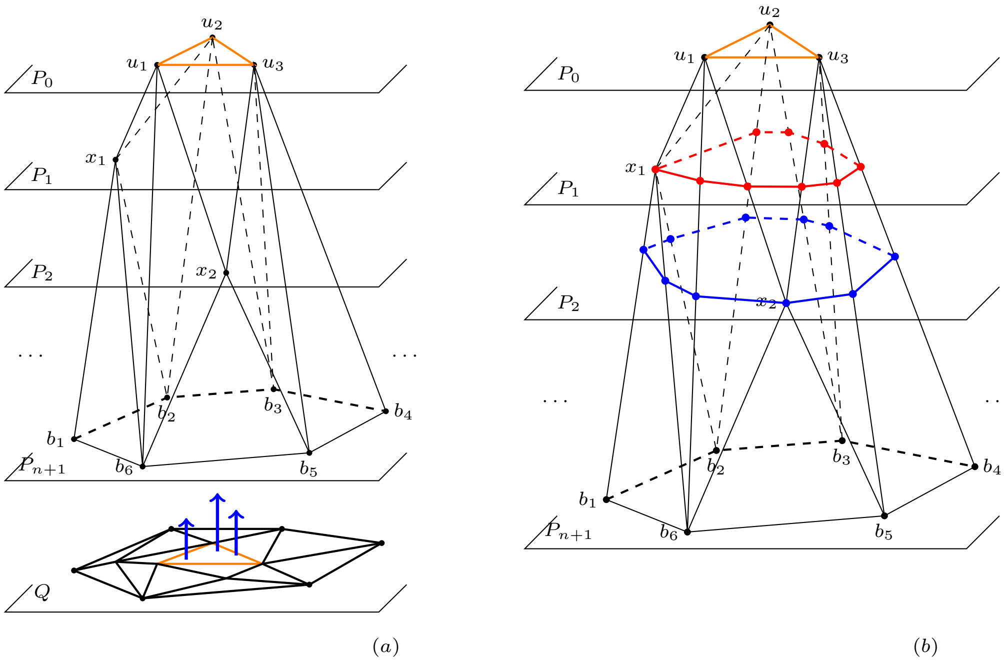

Fig. 12. The pyramid Π and the decomposition L' of the pyramid Π.

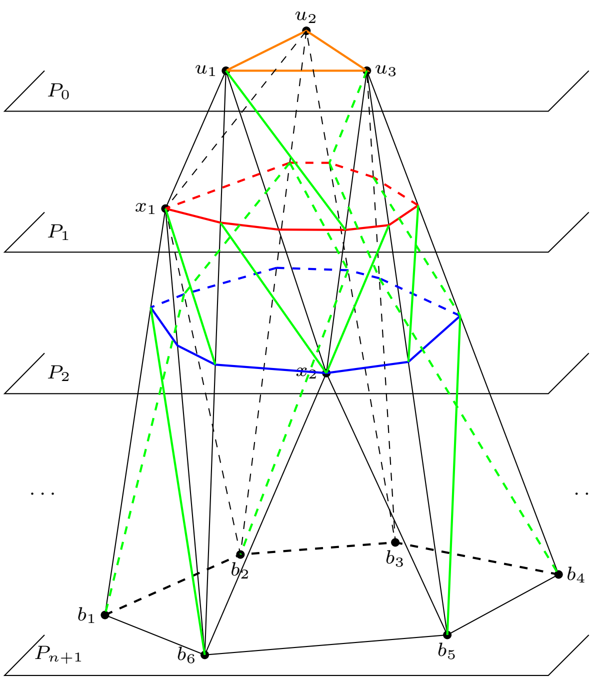

Fig. 13. The sub-triangulation L'' of the pyramid Π.

of K'' and L'', respectively. Moreover, in the subdivision L'', each edge is the common edge of exactly two triangles, and likewise in K''. That implies that K'' is a triangulation.

Now, we prove, by induction, that the intersection of L'' with each plane P_i is a curve homeomorphic to a circle. We see that $L \cap P_0$, which is the boundary of the triangle σ_0 denoted by B_0, is homeomorphic to a circle. Assume that B_i is homeomorphic to a circle but B_{i+1} is not homeomorphic to a circle. Then L''

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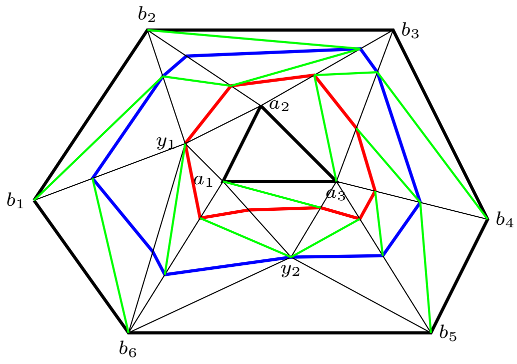

Fig. 14. The sub-triangulation K'' of the polygon.

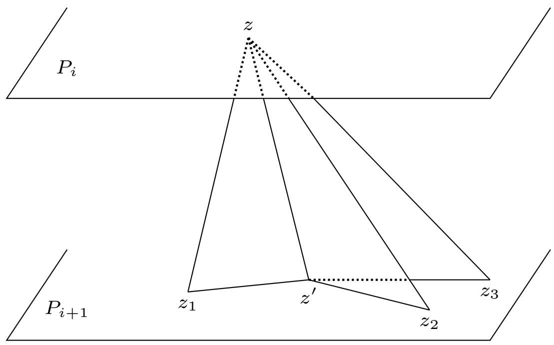

Fig. 15. Not admissible picture: The intersection B_i of L with each plane P_i can not have multiple points.

has multiple points in the plane P_{i+1} (see Fig. 15). By projection, K'' is no longer a triangulation. That provides a contradiction.

5) Step 5: Apply Cauchy's method (section 1.2): Now, let us apply Cauchy's method on the pyramid starting by removing the triangle σ_0. Assume that we have already removed all triangles above the plane P_i. We will prove that if we remove all the triangles in the band situated between P_i and P_{i+1}, the sum $n_0 - n_1 + n_2$ does not change. This fact can be established processed since the open band between B_i and B_{i+1} does not possess vertices, as follows: Let us fix a triangle $(\alpha_0, \alpha_1, \beta_0)$ of the band between B_i and B_{i+1}, where the vertices α_0 and α_1 belong to B_i and β_0 belongs to B_{i+1}. First, we remove the triangle $(\alpha_0, \alpha_1, \beta_0)$ by operation I, without changing the sum $n_0 - n_1 + n_2$. Now, the edge (α_1, β_0) is an edge of either the triangle $(\alpha_1, \beta_0, \beta_1)$, where $\beta_1 \in B_{i+1}$ (see Fig. 16 (a)), or of the triangle $(\alpha_1, \alpha_2, \beta_0)$, where $\alpha_2 \in B_i$ (see Fig. 16 (b)). In the first case, the triangle $(\alpha_1, \beta_0, \beta_1)$ can be removed by operation I of Cauchy's process, and in the second case the triangle $(\alpha_1, \alpha_2, \beta_0)$ can be removed by operation II. In both of these two cases, the sum $n_0 - n_1 + n_2$ is not changed.

We continue the process for all the triangles of the band, all of which are one of two cases above, until we reach the last vertices of B_i and B_{i+1} situated before going back to α_0 and β_0, respectively. We call these vertices α_k and β_j. We have two possible situations (a) and (b) (see Fig. 16). In situation (a), the last remaining triangles are $(\alpha_k, \alpha_0, \beta_j)$ and $(\alpha_0, \beta_j, \beta_0)$. In this case, these triangles can be removed in this order using operation II. In situation (b), the remaining triangles $(\alpha_k, \beta_j, \beta_0)$ and $(\alpha_k, \alpha_0, \beta_0)$ can be removed in this order using also operation II. In the two cases, the sum $n_0 - n_1 + n_2$ is not changed.

6) Step 6: The conclusion.

The process continues until we reach the boundary B_{n+1} of the hole, which is the boundary K_0 of K, and is also the intersection of L'' with the plane P_{n+1}.

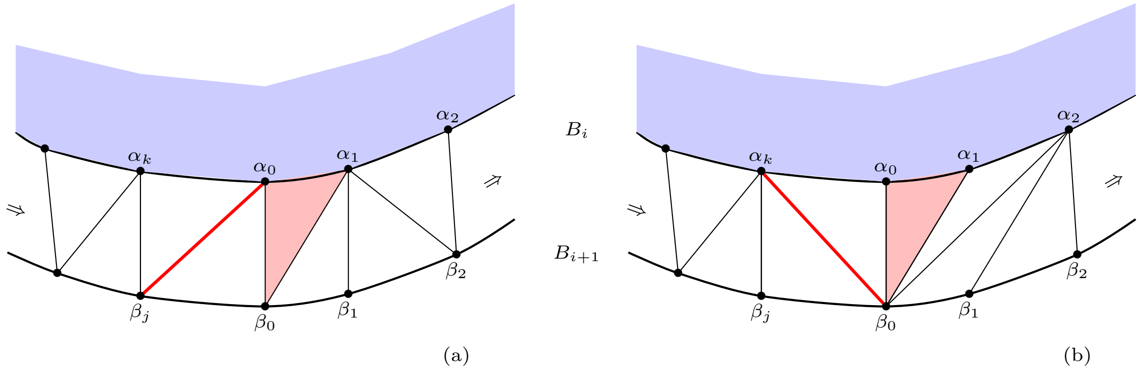

Fig. 16. Going from B_i to B_{i+1}.

Let us denote by n_0^K, n_1^K and n_2^K respectively the numbers of vertices, edges and triangles of the triangulation K and let us use the same notation for the sub-triangulation K'' and the triangulation K_0. We have

$$n_0^K - n_1^K + n_2^K = n_0^{K''} - n_1^{K''} + n_2^{K''} = n_0^{K_0} - n_1^{K_0} + 1.$$

The first equality follows from the fact that the sub-triangulation process does not change the alternating sum and the second equality comes from the fact that we removed the triangle σ_0 at the beginning and that $n_2^{K_0} = 0$. Here we take the identifications of simplices of K_0 into account.

Given a triangulation K of the polygon, the result does not depend on the choices made. □

Proof of Theorem 1.1 using Cauchy's method. Let $\widehat{K}$ be a convex polyhedron. We proceed with the planar representation K of $\widehat{K}$, according to the first step of Cauchy's proof (see Fig. 2). Notice that K is a polygon without any identification of simplices on its boundary K_0. Then $n_0^{K_0} - n_1^{K_0} + n_2^{K_0} = 0$. Theorem 2.1 implies that $n_0^{\widehat{K}} - n_1^{\widehat{K}} + n_2^{\widehat{K}} = +2$, taking into account the removed polygon $\mathcal{P}$ in the first step of Cauchy's proof. Since Theorem 2.1 is proved using only Cauchy's method, then Euler's formula is also proved by using only Cauchy's method. □

Before going further with applications, let us provide some remarks on the proof of Theorem 2.1.

Remark 2.2. The stereographic projection proof of Euler's formula is a particular case of the proof.

Remark 2.3. There are other ways to define an order of the vertices of the polygon to be able to draw the pyramid, without using the Euclidean distance in $\mathbb{R}^2$.

One possible way is the use of notion of distance between two vertices as the least number of edges in an edge path joining them. As in the proof of Theorem 2.1, choose a 2-dimensional simplex σ_0 in the interior of the polygon K and define distance 0 for the three vertices of σ_0. Determining any order between vertices whose distance to vertices of σ_0 is 1, one continues the ordering determining any order between vertices whose distance to vertices of σ_0 is 2, etc. Then one proceeds with the construction of the pyramid.

Another way would be to start the proof of Theorem 1.1 with the convex polyhedron and order the 2-dimensional faces according to the shelling process (see [34] and [2]). The 2-dimensional faces of the polyhedron are dual to the vertices of the polar polyhedron. One obtains an order on the vertices of the polar polyhedron. One continues the proof using the polar polyhedron instead of the original polyhedron, knowing that the sum $n_0 - n_1 + n_2$ is the same for the polyhedron and its polar. However, the shelling is a tool which was defined well after Cauchy's time, so it is not acceptable in our context. We mention it just for the sake of completeness.

Remark 2.4. In step 4 of the proof, the projection on the plane Q of the intersection of each plane P_i with the pyramid is a Jordan curve passing through y_i. Moreover, in step 5 of our proof, we make it clear that if the boundary of the extended hole is homeomorphic to a circle, then Cauchy's process works. It is then possible to proceed in step 5 either with the sub-triangulation L'' of the pyramid or with the sub-triangulation K'' of the polygon K.

Remark 2.5. In step 5 of the proof we use only the operations I and II of Cauchy. Observing that if we change the order of removal of the last remaining triangles, for example, in situation (a), if we remove the triangle $(\alpha_0, \beta_j, \beta_0)$ and thereafter the triangle $(\alpha_k, \alpha_0, \beta_j)$, we will use first operation I of Cauchy and then the operation that we called operation III in section 1.3 (see Fig. 6 (d)). Here also, we do not change the sum $n_0 - n_1 + n_2$.

Remark 2.6. As we emphasized at the beginning of this section, we use in the proof only Cauchy's method (section 1.2) without other tools. We know very well that there exist "modern and faster" ways to prove Theorem 2.1. However, these proofs use tools that appeared after Cauchy's time, in particular some proofs use the Jordan Lemma, which, as we have seen, appears as an artifact in our proof.

3. Applications

In the following, using Theorem 2.1 as the main tool, we prove that the sum $n_0 - n_1 + n_2$ does not depend on the triangulation in the case of the sphere, the torus, the projective plane, the Klein bottle and even for a singular surface: the pinched torus. In each case, we use a planar representation of the surface homeomorphic to a disc with possible identifications on the boundary, and the following lemma concerning the "cutting surfaces" technique that was introduced by Alexander Veblen in a seminar in 1915 (see [1]). This idea is well developed in the book by Hilbert and Cohn-Vossen [18], in particular for the surfaces that we give as examples.

The following "cutting" lemma will be used in the forthcoming proofs.

Lemma 3.1. *Let T be a triangulation of a compact surface. Let Γ be a continuous simple curve in S. There exists a sub-triangulation T' of T, with curvilinear simplices, compatible with the curve (that means Γ is an union of segments of T') in such the way that the number $n_0 - n_1 + n_2$ is the same for T and T'.*

Proof. First of all, we can assume that the curve Γ is transversal to all the edges of T, *i.e.* the intersection of Γ with each edge is a finite number of points. Otherwise, a small perturbation of Γ allows us to obtain transversality.

We choose a base point (the starting point) x_0 on the curve, as well as an orientation of the curve. If the curve is not closed, we define the base point as one of the two extremities. The following process does not depend on either the starting point, or the orientation of the curve.

A sub-triangulation T' is built simplex by simplex following the orientation of the curve Γ. The first subdivided simplex is the one σ_0 containing the base point. Let y be the first point where the curve leaves σ_0. The (curvilinear) segment (x_0, y) will be an edge of T' as well as segments connecting x_0 to vertices of σ_0, one of which can be (x_0, y) if the point y is a vertex of σ_0 (Fig. 17).

Now, it is enough to perform the construction for a simplex $\sigma = (a, b, c)$ the curve Γ enters. In the following construction, we assume that all the simplices that the curve meets between the base point and the simplex σ (with the given orientation) are already subdivided. The "entry" point of Γ in the simplex σ can be either a vertex or a point d located on an edge of σ.

If the entry point of the curve Γ is a vertex a, the curve can exit at a point d located either in the opposite edge, or in an edge containing the vertex a, or at another vertex. In the first case (Fig. 18 (i)), we

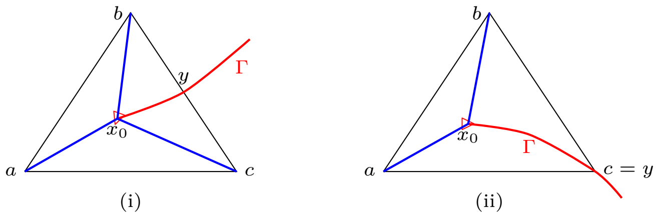

Fig. 17. Subdivision of the first simplex I.

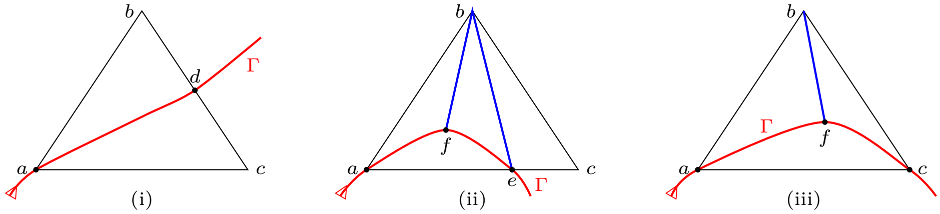

Fig. 18. Subdivision II.

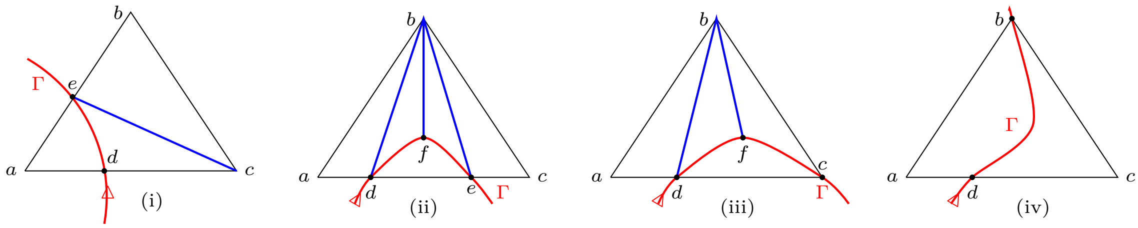

Fig. 19. Subdivision III.

divide the triangle (a, b, c) into two (curvilinear) triangles (a, b, d) and (a, d, c). In the second case (Fig. 18 (ii)), let $e \in (a, c)$ be the point at which the curve Γ exits from the triangle and we choose a point f on the curve, located between a and e. We divide the triangle (a, b, c) into four (curvilinear) triangles (a, b, f), (b, f, e), (b, e, c) and (a, f, e). Finally, in the last case (Fig. 18 (iii)), assume that the exit point is the vertex c, we choose one point f on the curve, located between a and c. We divide the triangle (a, b, c) into three (curvilinear) triangles (a, b, f), (b, f, c) and (a, f, c).

Notice that, in all the three cases, the choice of sub-triangulation is not unique, but the sum $n_0 - n_1 + n_2$ remains unchanged independent of the choice.

If the entry point of the curve Γ is located in one edge, we denote by d the entry point and by (a, c) the edge of σ containing d. The next exit point of Γ can be either in an edge different from (a, c) (for example (a, b)), or in the same edge (a, c), or it can be a vertex (see Fig. 19).

In the first case (Fig. 19 (i)), we define, for example, a sub-triangulation of the triangle (a, b, c) formed by the (curvilinear) triangles (a, d, e), (c, e, d) and (c, e, b).

In the second case, for example if the curve Γ enters and exits by two points d and e situated on the same segment (a, c), we choose a point f on the curve located between d and e (Fig. 19 (ii)). We define a sub-triangulation of the triangle (a, b, c) formed by five (curvilinear) triangles (a, b, d), (b, d, f), (b, f, e), (b, e, c) and (d, f, e).

The two last cases of Fig. 19 (iii) and (iv)) are similar to the cases of Fig. 18 (ii) and (i) respectively.

In the four cases, the choice of sub-triangulations is not unique, but the sum $n_0 - n_1 + n_2$ remains unchanged independent of the choice.

The process continues for all the 2-dimensional simplices crossed by the curve Γ, and it is completed in a finite number of steps because the number of simplices is finite, even after the subdivision. $\square$

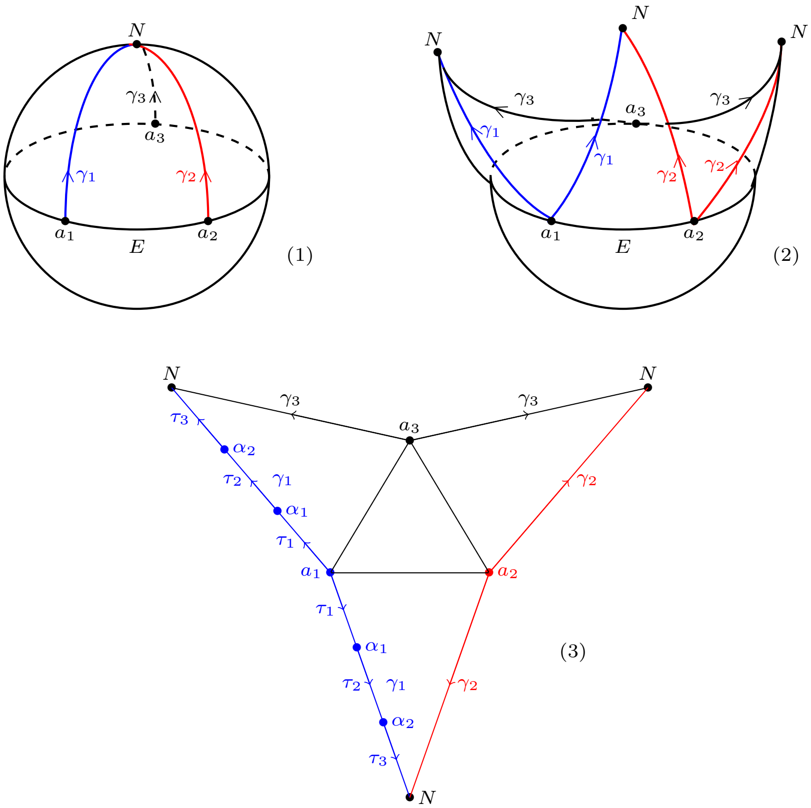

Fig. 20. Planar representation of the sphere.

3.1. The sphere case

In the following, we provide an alternating proof of Euler's formula for the sphere using Theorem 2.1 as the main tool, with the idea of "cutting surfaces" as an additional tool.

Let T be a triangulation of the sphere $\mathbb{S}^2$. We consider four curves on the sphere: The equator E (or any parallel) and three curves γ_1, γ_2 and γ_3, going from the North pole N to the curve E along meridians. Let us denote by a_i the intersection points $\gamma_i \cap E$, where $i = 1, 2, 3$. Using Lemma 3.1 we can construct a subdivision T' of the triangulation T compatible with the four curves, *i.e.* such that the union of the four curves is a subcomplex of T'. By Lemma 3.1, the sum $n_0 - n_1 + n_2$ remains the same.

Now, we cut the sphere along the curves γ_1, γ_2 and γ_3, in such a way that the projection provides a polygon K in the plane containing the equator (see Fig. 20). Notice that the projection of T' gives us a sub-triangulation K' of K. We obtain a planar representation of the sphere, homeomorphic to a disc with identifications of the simplices on the boundary K_0 of K corresponding to the cuts.

Theorem 2.1 states that the sum $n_0^T - n_1^T + n_2^T$ of the triangulation T is equal to the sum $n_0^{K_0} - n_1^{K_0} + 1$, where the sum $n_0^{K_0} - n_1^{K_0}$ is calculated by the boundary of the figure. Notice that, using the same notation on the sphere and on the planar representation, the vertex N is common to all the curves γ_i and must be identified. Beside this vertex N, the number of vertices in each curve γ_i is equal to the number of edges (see Fig. 20). Then for the boundary of the planar representation, we have $n_0^{K_0} - n_1^{K_0} = +1$ and for the triangulation, we have

$$n_0^T - n_1^T + n_2^T = +2.$$

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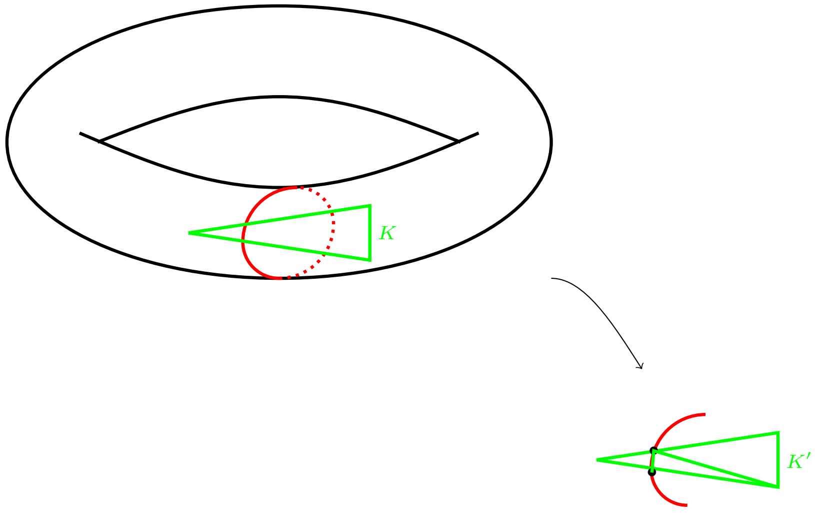

Fig. 21. Sub-triangulation T' of T.

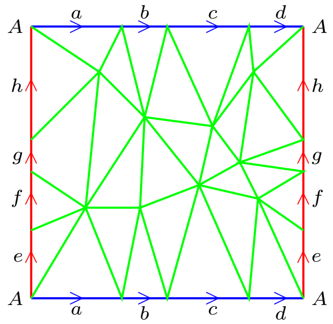

Fig. 22. A planar representation of torus.

3.2. The torus case

Let T be a triangulation of the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$. We choose a meridian $M = \mathbb{S}^1 \times \{0\}$ and a parallel $P = \{0\} \times \mathbb{S}^1$. They cross at one point $A = \{0\} \times \{0\}$. Observe that, without loss of generality, we can choose them transversally to all the edges (1-dimensional simplices of T). We define a sub-triangulation T' of T, in the following way (see Fig. 21): Each triangle σ (2-dimensional simplex) of T meeting M or P is divided in such the way that $\sigma \cap M$ (or $\sigma \cap P$) is an edge of T'. Lemma 3.1 implies that the sum $n_0 - n_1 + n_2$ remains the same for T and T'.

Now, cutting the torus along M and P, we obtain a planar representation K of the torus which is homeomorphic to a disc, with identifications on the boundary K_0 corresponding to the cuts (see Fig. 22). Hence, by Lemma 3.1, the number $n_0 - n_1 + n_2$ remains unchanged. Using Theorem 2.1, we have:

$$n_0^T - n_1^T + n_2^T = n_0^{T'} - n_1^{T'} + n_2^{T'} = n_0^K - n_1^K + n_2^K = n_0^{K_0} - n_1^{K_0} + 1.$$

Now, with the identifications on the boundary K_0, we have $n_1^{K_0} = n_0^{K_0} + 1$. Finally

$$n_0^T - n_1^T + n_2^T = 0$$

for any triangulation of the torus.

The same proof holds for the torus of genus g. For example, let us take a torus of genus 3. Given a triangulation T of the torus, we fix a point A and around each “hole” of the torus, we fix a “meridian”

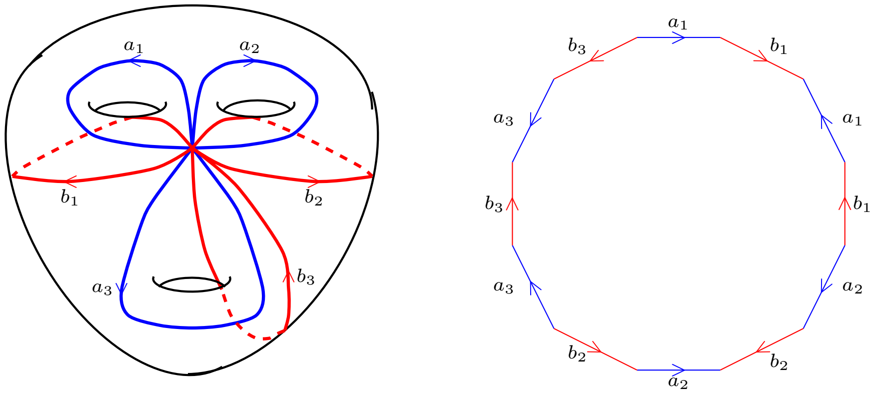

Fig. 23. Planar representation of the torus of genus 3. In order to have a light figure, we did not draw a triangulation T' in such a way that each edge a_i and b_i are subdivided in at least three segments.

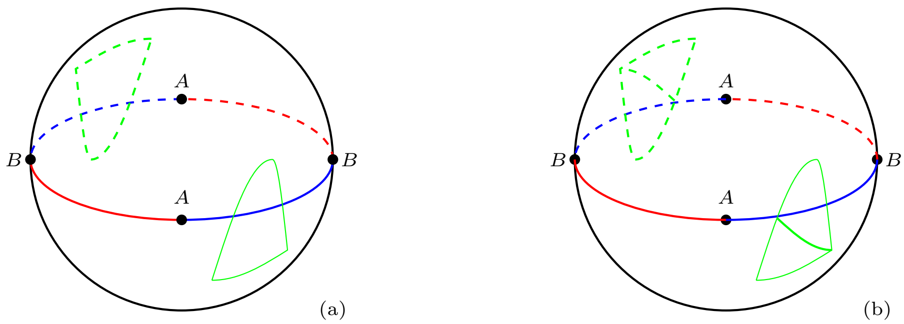

Fig. 24. Triangulation T of the projective plane. Sub-triangulation T' of T.

(a_1, a_2, a_3 in Fig. 23) and a "parallel" (b_1, b_2, b_3 in Fig. 23). We construct a sub-triangulation T' of T by the same methods as in the case of the torus. Cutting the torus of genus 3 along the meridians and the parallels, we obtain a planar representation of the torus of genus 3, triangulated by the triangulation T'. By the same procedure as in the torus case, one obtains $n_0 - n_1 + n_2 = -4$. This is an example of Lhuilier's formula (1.4).

3.3. The projective plane case

The projective plane is represented by a sphere whose diametrically opposite points are identified. A triangulation of the projective plane is given by a triangulation of the sphere which is symmetric with respect to the center of the sphere. Let us consider the sphere in $\mathbb{R}^3$ (see Fig. 24 (a)) and let T be such a triangulation of the projective plane.

The intersection of T with the equator defines a triangulation J of the equator that is symmetric with respect to the center of the sphere. Let us define a sub-triangulation T' of T such that simplices of J are simplices of T' and such that T' is symmetric with respect to the center of the sphere (see Fig. 24 (b)). By the Lemma 3.1, the sum $n_0 - n_1 + n_2$ is the same for T and T'.

Now, the orthogonal projection of the northern hemisphere to the plane $0xy$ provides a triangulation of the disc D of radius 1, centered at the origin, whose triangulation of the boundary is symmetric with respect to the center of the disc. With the identification of simplices, we have $n_0 - n_1 = 0$ on the boundary (Fig. 25). Then, by Theorem 2.1, we have

$$n_0^T - n_1^T + n_2^T = +1$$

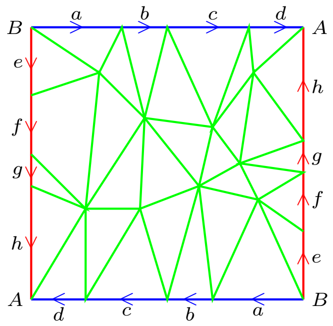

Fig. 25. A planar representation of the projective plane.

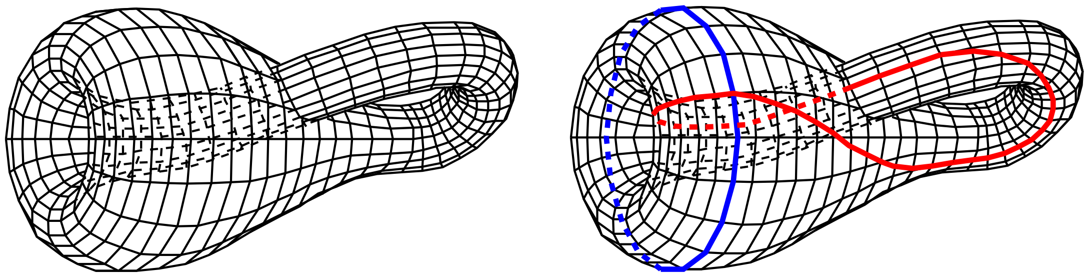

Fig. 26. The Klein bottle and cuts (the meridian M is in blue and the parallel P is in red).

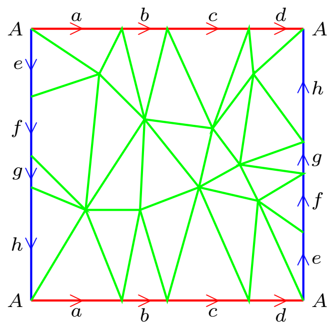

Fig. 27. A planar representation of the Klein bottle.

for any triangulation of the projective plane.

3.4. The Klein bottle case

The case of Klein bottle is similar to the case of torus. Given a triangulation T of the Klein bottle, we choose a meridian M and a parallel P (see Fig. 26). Let us define a sub-triangulation T' of T compatible with M and P. The cut along M and P provides a planar representation of the Klein bottle as a rectangle triangulated with identifications on the boundary (see Fig. 27). On the boundary, we have $n_1 = n_0 + 1$. Then, by Theorem 2.1, we have

$$n_0^T - n_1^T + n_2^T = 0$$

for any triangulation of the Klein bottle.

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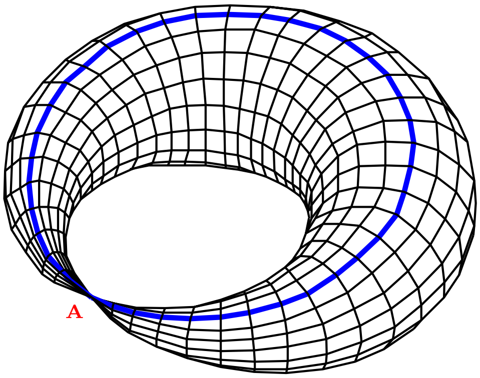

Fig. 28. The pinched torus and cut.

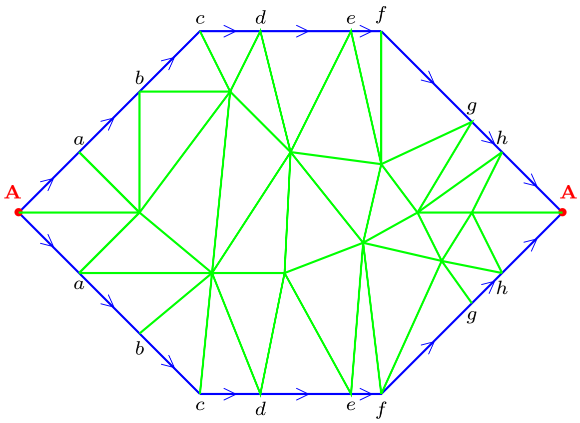

Fig. 29. A planar representation of the pinched torus.

3.5. The pinched torus case

Not every surface with singularities admits a planar representation. The pinched torus is an example of a singular surface that does admit such a planar representation.

Let us recall that the pinched torus is a surface in $\mathbb{R}^3$ defined by the following cartesian parameterization:

$$\begin{cases} x = \left(r_1 + r_2 \cos(v) \cos\left(\frac{1}{2}u\right)\right) \cos(u) \\ y = \left(r_1 + r_2 \cos(v) \cos\left(\frac{1}{2}u\right)\right) \sin(u) \\ z = r_2 \sin(v) \cos\left(\frac{1}{2}u\right) \end{cases}$$

where r_1 and r_2 are respectively the large and small radii (see Fig. 28).

Let T be a triangulation of the pinched torus. We choose a “parallel” P passing through the singular point A of the pinched torus and we define a sub-triangulation T' of T compatible with P using Lemma 3.1. By cutting along P, one obtains a planar representation of the pinched torus (Fig. 29) with identifications on the boundary. One observes that the point A is duplicated. On the boundary, we have $n_0 - n_1 = 0$. Then, by Theorem 2.1, we have

$$n_0^T - n_1^T + n_2^T = +1$$

for any triangulation of the pinched torus.

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